Lorentz and the hyperbola

Or: why didn’t you say that right away?

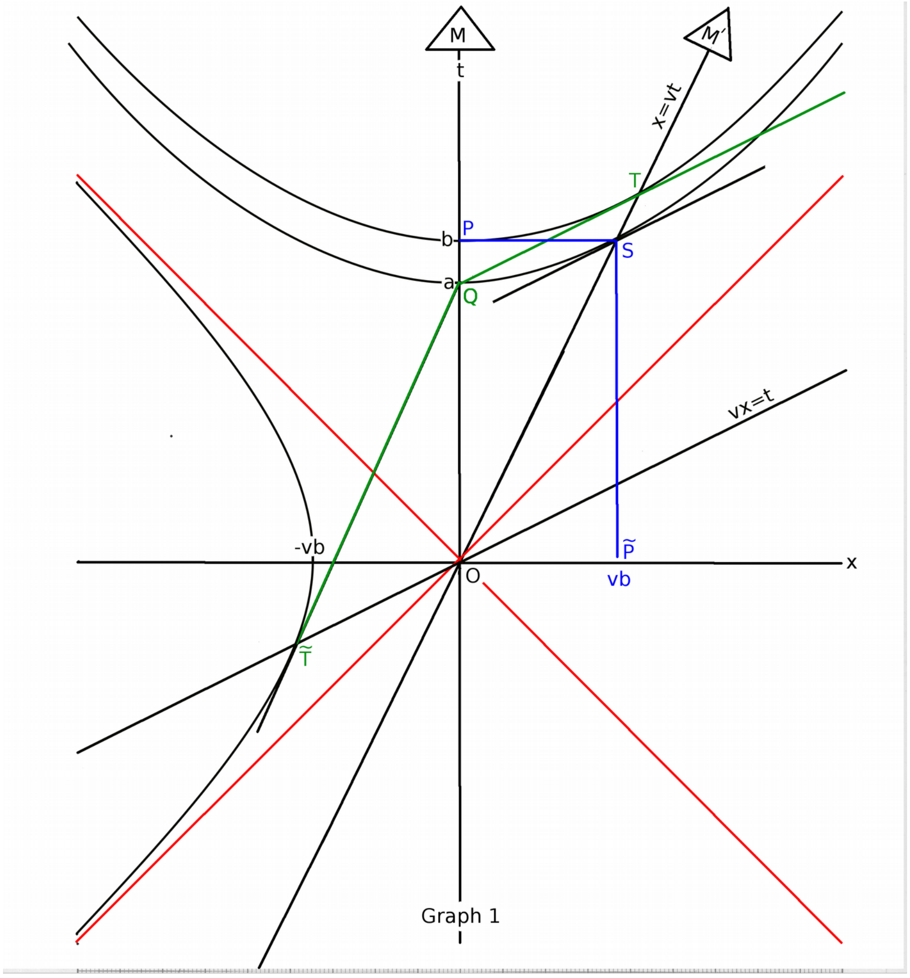
Why this?

I saw quite a number of introductions to special relativity while trying to understand the subject but never found it explained as I now do to myself. I would think it turns the subject into high school stuff, not even surely of the final year.

Linge River, Netherlands, November 2016,

Introductory overview

In special relativity theory it got accepted that if M and M´ move relative to each other, their meters and seconds will deform if expressed in each other. This implies you have M-meters, M-seconds, M´-meters and M´-seconds. M and M´ will have different meter-second grids, and the Lorentz-transformation translates those grids in each other.



The standard graph to first illustrate this is the one dimension relativistic space-time diagram. All movements are *uniform*: there is no force (like gravity or electromagnetism) acting upon any mover. "One dimension" means we only move along one line, either in the negative or in the positive direction, that is leftward or rightward along the *x* (distance) line in *Graph 1*. The successive moments in time of this process are stacked vertical upward along the *t* (time) line.

Time measurements by M are on vertical axis, distance measurements by M on the horizontal axis, we, the onlookers, made M's grid of space-time measurements the orthogonal grid. We consider movers and two light pulses that come together at $(x,t)=(0,0)$ and then start moving away from each other from there. But there is a *past* as well. That is the lower half of the graph where t is negative. In that past all this moving stuff approaches the origin to come together and pass each other at its different speeds exactly at $(x,t)=(0,0)$.

This must have been done in the infinite past (- ∞, down in the graph) by absolute master shooters: first they launched, from the far left to the right over the x-axis, M, then, after exactly the appropriate time, they launch M´ such that M´ once out of acceleration and in uniform motion, has a speed relative to M of v exactly. Then, after again some waiting, they shoot two light pulses, one from the far left after M and M´ to catch up with them, and one from the far right to go for a frontal meeting, *all such that after the time would have lapsed from - ∞ to 0 all four are exactly (infinitesimal!) together in* $(x,t)=(0,0)$. We should appreciate their incredible performance : it makes our space-time diagram analysis easy and instructive.

As usual I normalize the time and distance scales to set light speed $c \equiv 1$. This sets in the graph the line-length of a light-year equal to the line length of a year, or for instance, in micro-physics, the length of a light-nanosecond (=30cm) equal to a nanosecond. Years or nanoseconds, that is a matter of zooming the graph in and out, but that will not change the slope of the linear time path of the light pulses, that is, the series of *events* (*x*,*t*) of distances and clockpositions where the light pulse will be at. These lines are called the *worldlines* of the two light pulses. They form the red cross in *Graph 1*. Once the graph is normalized, one light worldline slopes 45° (for the left-to right-pulse, moving in same the direction as M and M'). The other light worldline, the one for the pulse traveling in the opposite direction of M and M' and of the other light pulse, slopes 45° to the left up.

More about normalization in section 3

Another way to state this: we onlookers chose ("set") M in *Graph 1* as the "zero-mover". The speed of any other uniform mover M´ will be drawn in space-time diagrams graphs like *Graph 1* as

$x=vt$ (2)

where (*x*,*t*) the speed *v* at which M´ moves from M (*v* is negative if M´ moves from M in M's negative direction). Thus a mover speeding away from M at the origin O with *v*=50% of light speed will, in M's graph, move along a worldline of $x=0.5t$. And *x* is defined as the distance of M´ from M. In the graph, M´ 's path in time is OST. M's worldline is, by definition, OQP, where at all times *t*: $x=0$. Speed *v* can be read in *Graph 1* as

$$v= d(PS) \big/ d(S\tilde{P})$$

where *d*(PS) is the notation for "distance between P and S", so if P is $(x,t)=(0,b)$ then $\tilde{P}$ is $(x,t)=(vb,0)$. The maximum rearward speed is $-c \equiv -1$ (lightspeed), the maximum forward speed is $c \equiv 1$, so for moving hardware speed is limited to $-1<v<1$.
The points of the graph are *events* of the type: "M's clock reads *t*", or as we shall phrase it everywhere: *mover M's clockposition is t*. To distinguish between the positions *t* of different movers' clocks, we shall use *t* for M and *t´* for M´. So in our example, we have two movers, thus two *t*-scales. There are no t-scales apart from those of movers, since there is no absolute time. The universe has no time, only hardware in the universe has, and movers at different speeds experience the clockpositions on each other's scales as *different from their own*. The Lorentz equations make sure we can convert those scales into each other (section 10).
Every point in the *Graph 1* could be an *event* of the type "M's clockposition is t". For that to be so, some actual mover's specific value of speed *v* should be such that his worldline passes through that point. In *Graph 1*, on the vertical axis are all *clockposition events* of M, and on the line *x*=vt you find all *clockposition events* of M´.

When starting to study the geometrics of special relativity last year, it soon dawned on me that I was not sure about what exactly is a "clock". None of my introductory texts was very explicit about it. I saw a lot illustrations sporting round things that should remind me of an analog revolving display of what looked most like the type of clock that works on the balance wheel with a spring. After some days of halting at the matter I found that *any regular process in nature can be used as a clock*: the seasons clock the year, the revolving sun clocks the day. In Antiquity, Roman prostitutes used a slightly leaking pottery bowl floating on water. It gave a discrete sound when sinking to the bottom of its vessel. At the same time came the sand glass, and it considerably improved through the ages. Then man's ambitions to progress yielded the pendulum-clock, the balance wheel with spring, cesium radiation counters. Did we really improve our "clocks"? This is answered by comparatively testing clock types: you set a sizable series of each type of clock to zero at some moment, after a while read them at some other moment, then check the deviations within the group of sand glasses, the group of pendulums, the group of cesium counters. Your conclusion will have certainty only by a statistical margin, but in this case be sure your ranking will be beyond your reasonable doubt, and you will rank top the cesium counter, even proving that two earth days always slightly differ.
A clock, a regular iterative process that is, *is a tool used by most creatures on earth that maintained themselves in evolution, including trees, wasps and human beings.* Apart from their biological clock, humans from the dawn of the species use ever more sophisticated clocks (starting with the day-and-night cycle now having reached the cesium counter) on which you can react to do whatever you need to in order to survive. In speculative physics it is assumed that less regular processes waver in rough accord with

the regular ones. This for instance is assumed in the twin paradox, where twins start to deviate in “age”. In this “paradox”, body “aging” of human primates is supposed to move with the speed of the “clock” they carry with him when on the move. That is speculative indeed, for the real experiment is still impractical, but I personally believe this rough accord will once prove to exist in much the same way as the revolution time of the earth does not waver way out of step with the cesium counter clock and tends to an average slightly dropping through the ages. The predicted exact day length for today Thursday, 17 November 2016, is 24 hours, 0 minutes and 1.2583 milliseconds. For yesterday it was 24 hours, 0 minutes and 1.4101 milliseconds.
In our space-time diagram we have two worldlines, the vertical one of M and x=vt for M´. Strictly we do not have *all* points on those two lines as events, but only a subset, since we shall handle the graph as mathematically infinitesimal and “clocks” are not: they have a maximum resolution, there will always be time between two counts of whatever clock you make, so we do not have an infinite number of events. But the points of events that are really measured and logged should be on those lines.

Now consider hyperbola *a* in the *Graph 1*. Its algebraic expression is (explained in sections 1 and 2):

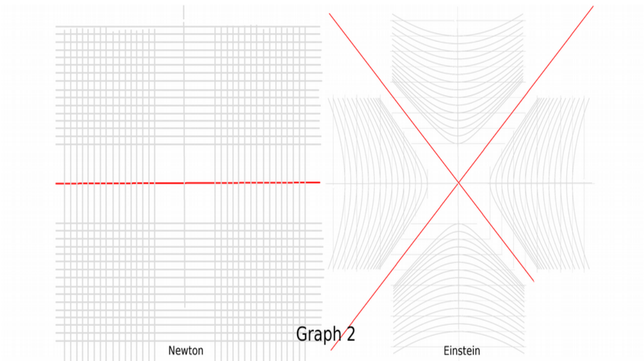
$x^2 - t^2 = -a^2$

Consider for a moment the infinite set of movers of all speeds, all real values $-1<v<1$, that is all worldlines through O. Then hyperbola *a*, $x^2 - t^2 = -a^2$, is the curve connecting the points of all events where one of these movers’ clockposition is *a*. Or: every mover’s clockposition event *a* is at the point where its worldline cuts through the *a*-hyperbola. If we had movers of all speeds, those intersection points together would form the *a*-hyperbola. That, oddly enough, is an empirical law, first found in electrodynamics, later found to hold generally. We have drawn only two movers, M and M´, thus only two such “clockposition=*a*” events: Q and S. In point Q, mover M ’s clockposition is *a*. In point S, mover M´ ’s clockposition is *a*.

Hyperbola *a* is *time frontier a*: under it, all movers’ clockpositions point *before a*, above it, all mover’s clockpositions point *after a*.

Thus as shown in *Graph 1*, in M ’s log, the event of point S “M´ ’s clockposition is *a*” is clocked at the M’s event of point P, that is on a clockposition different from a. We call it *b*. Strictly put: M measures the event S: “M´ ’s clockposition is *a*” as happening *simultaneous with* (“at the same time as”, “*equitemporal* to”) its home event P where its own clockposition is *b*. Thus, our zero-mover M measures the clock event “M´ ’s clockposition is *a*” as *later* than its own clockposition event Q: “my clockposition is *a*”. The time difference (time dilation) is PQ in the graph. At this stage we only know the value of this time difference *b-a* as *measured by M*.

In Newtonian mechanics, you don’t have hyperbolas. You have horizontal lines. Light speed is implicitly assumed to be infinite so a Newtonian worldline of light (*avant la lettre*, these graphs got invented by Minkovski in 1907) would be the horizontal axis. Things turned out to be not like that, and in *Graph 2* you see that the error gets worse with larger relative speeds v (going to all corners). The full meaning of these comparative graphs will be clear only after section 12.



Graph 2

Read in Graph 1: In M ’s measurement, the event of point S: “M´ ’s clockposition is a” is simultaneous *to the event of point P: “M ’s - my - clockposition is b”, and* later *than Q: “M ’s - my - clockposition is a”*

Now we do the reasoning from the other side. This is more difficult since we chose to give M the vertical time axis, and the horizontal distance axis. M ’s measurement system is set orthogonal. That is easy graph reading since events (points in the graph) are, for instance, b-simultaneous to M if they are on the horizontal line through b. And all parallel vertical line have all points (events) with the same distance to M *as measured from M*. That means that all points on, for instance, $\widetilde{PS}$ in *Graph 1* all have the same distance *vb* to M, *measured from M*.

But measuring from M´, simultaneity and equidistance must be read differently in the graph. We drew two hyperbolas: *a* and *b*. Here, we think of a and b as fixed values so let us write this infinite parameter family of hyperbolas for arbitrary values $\tau \in \Re$ as

$x^2 - t^2 = -\tau^2$, so τ can be any real number, and we think of *a* and *b* as two such real numbers.

Both M and M´ experience events like "my clock reads *a*", "my clock reads *b*", and this how each of them defines his time, called "home time" or "rest time". But M 's and M´ 's home times differ. Worse: the difference is measured differently from the two sides. Even worse, there is no compromise to strike, these are two standpoints that cannot be turned into one. Both can measure all events, both "home"-events and "away"-events, using their own home time and home distance as the grid. We did so for M, now we have to shift to reading M´ 's time and distance grid. After we have the transformation, both movers can of course translate their data in terms of the other's measuring grid.

Think of British and Egyptian pounds. Their value differs. Whoever plans an investment relating the UK and Egypt can calculate and write his project sheet either in British pounds or in Egyptian pounds, depending upon whether he mainly deals with British or Egyptian oriented partners. Or he can print a project sheet transformed into the other currency in an appendix, for those to whom this is more convenient. Similarly we shall now have M-meters, M-seconds and M´-meters. M´-seconds. The difference between them derives from relative speed *v* of M and M´.

Let us first consider local time in a small neighbourhood around M´. If M´, while being at point T, where his clock reads *b*, looks to movers with slightly more and slightly less speed, in what direction in the graph will you find the events where M´ 's near neighbours' clock also reads b? Those "clockposition *b*" events of a small section of movers near around T with only slightly different speeds should be be *along the b-hyperbola* $x^2 - t^2 = -b^2$. But there, locally, that hyperbola slopes seriously up to the right, down to the left. M would certainly *not* measure such a line section of heavily sloping equal clock events as happing on the same time. But M´ and his close neighbours do, in fact perceive the similar neighbourhood of M as equally heavily sloping.

Here we need some mathematical properties of the hyperbola generally. The slope d*x*/d*t* of a hyperbola $x^2 - t^2 = -\tau^2$ (for any value of τ) at the point where the worldline $x=vt$ of some mover with some speed v cuts through this hyperbola turns out $dx/dt=1/v$ (see section 6). This slope indicates the local time frontier at some point on an arbitrary worldline $x=vt$. If M´ 's worldline is $x = .5t$ (80% of light speed) this is $dx/dt=1/.5=2$.

Yes that is twice the speed of light, impossible for moving hardware. But it is no moving hardware. It is the local time frontier. More in section 5.

This value 1/*v* for local time frontiers where $x=vt$ cuts through the hyperbolas hold for the entire family of τ-hyperbolas, so everywhere along the worldline $x=vt$. This means that the tangential lines where $x=vt$ cuts through one of the hyperbolas of the family $x^2 - t^2 = -\tau^2$ *are all parallel.* The parallel lines through S and T in our graph are two examples. But! *This holds for any other mover's worldline with any other value of v.* In other words: for all *v* and all τ, at the point where $x=vt$ cuts though hyperbola $x^2 - t^2 = -\tau^2$, the hyperbola has $dx/dt=1/v$. (this proof, very important to the space-time diagram, is reproduced in section 6)

In that small neighbourhood around T, the iso-clockposition line is approximated by straight line through T with the slope 1/*v* , (if v=.5 it is 1/.5=2). In other words: it is approximated by the tangential line of the hyperbola $x^2 - t^2 = -b^2$ at T. That is the line through QT.

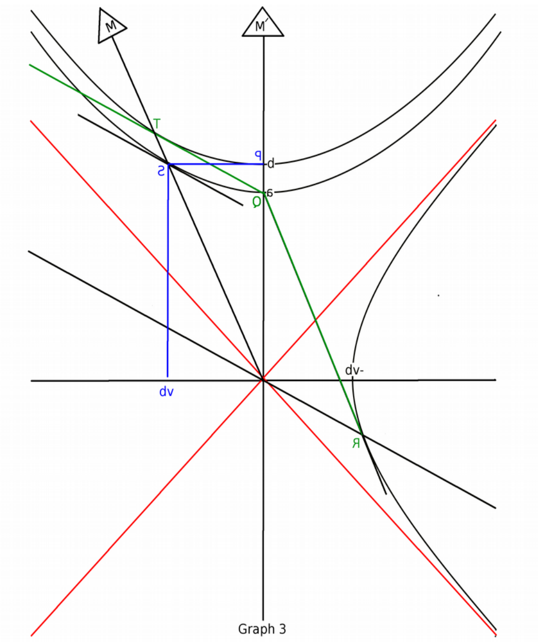
Yes, this implies that for instance at S, but generally at every point of M´ 's worldline, the tangential line of the τ-hyperbole which the worldline cuts through at that point has exactly the same slope 1/v. (more in section 6)

In our example, in M 's measurement, the event of point S: "M´ 's clockposition is *a*" is simultaneous to the event of point P: "M 's - 'my' - clockposition is *b*". In other words: points P and S, connected by the blue line, are *M-simultaneous*.
In our space-time diagram this should mean that the reverse holds as well: in M´ 's measurement, the event: "M 's clockposition is *a*" - but this is point Q! - is simultaneous to the event: "M´ 's - 'my' - clockposition is *b*" - which is point T . Again, in other words: points T and Q, connected by the green line, are M´-simultaneous (simultaneous in M´ 's measurement). (Proven in section 9)

In section 9 it is proven that this is a general mathematical property of the hyperbola family with parameter τ: $x^2 - t^2 = -\tau^2$. So this holds for every value τ , that is for the entire family of τ-hyperbolas, not only b. Hence any value for *b*, together with M´ 's worldline will give you *a* (as a function of *b*), and the event "M 's clockposition is *a*" will, a matter of pure math, be M´-simultaneous to "M´ 's – my – clockposition is *b*" (section 9).

So both movers M and M´ consider the moment on which the other's clockposition is *a* to be simultaneous to the moment their own clockposition b. That is the symmetry of relative time.

Another way to see the symmetry is to compare our graph with one where M´ 's clock and distance log is set orthogonal, so treated as the non-mover, or reference. This amounts to a crude image editor's horizontal flip of *Graph 1* into *Graph 3*.



This flipped graph presents the same relative movement as M moving *backward* from M´ with speed *v*= *minus* 0.5*c*. And, though in this version of the graph M´ is the one who has the directly readable values and the algebra of the measurement of M will be the hard one requiring consultation of the coming sections, all conclusions will be the same.

When there are different perceptions of simultaneity there must also be different perceptions of distance, for by "distance" between moving hardware we mean distance between two such things *at some moment*. Now two movers measure, as soon as they have a non zero relative speed *v*, different lines of events as simultaneous ("at the same moment") they take *different pairs of events* to measure the distance of the hardware involved, hence surely measure different values.

Distance measurement by M and M´, though yielding numerically different results, should be symmetric, just like time measurements. And will prove to be in section 8. M-meters and M´-meters differ as soon as M and M´ moves relative to each other, and it's quite analogous to the difference between M-seconds and M´-seconds. In *Graph 1*, a left-right *x*-hyperbola of equidistance measurement *x*=*vb* is drawn, to see that analogy. More about this in section 8.

The consequence, odd but welcome, that even inspired the very construction of the system we now use to compare measurements of different movers is that *the speed of light moving from* M *to* M´ *or vice versa will always be measured* ***by both*** *as c*, which we set in our graph as the unit of measurement. In meters and seconds it is 299 792 458 $msec^{-1}$.

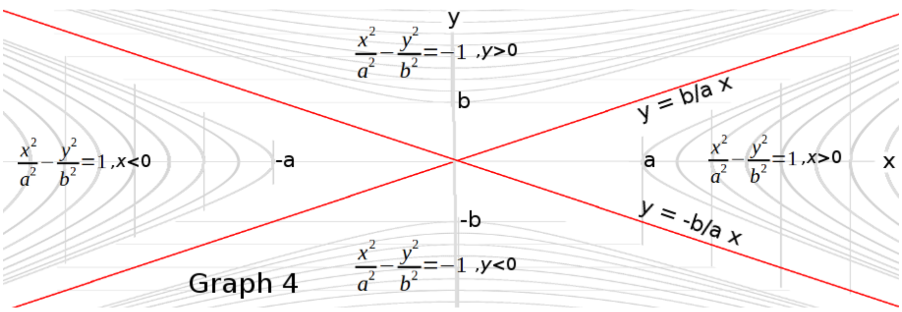
This can be practiced by launching light pulses from M ’s worldline to M´ ’s worldline and back. In the reference mover’s orthogonal grid they appear as 45° (up rightward c=1 or leftward c=-1). If you correct for the skewed grid of the other mover (see appendices) you will also find c=1 or c=-1. And in the flipped *Graph 3* that skewed grid is transformed into an orthogonal one and you directly see that the angle of these red light lines of the skewed grid indeed do not flip with the rest.

We can draw worldlines of light pulses. We did so for the special case of the two passing forward and backward at O. Those are the red lines. And we extended these two red lines into the third and fourth quadrant where everybody’s t is negative, where those two light pulses came from in the past, before reaching (*x*,*t*)=(0,0). But though light pulses have worldlines, this does *not* mean that in these calculations we can treat a light pulse as one of the two movers. For movers (hardware, with a mass) v=1 is an unattainable speed limit. The reason should remind of infinity (∞), which cannot be treated as a real number. Whatever τ you choose, even its limit to infinity, for a hyperbola $x^2-t^2=-\tau^2$ to reach the worldline of light, *x* and *t* should become infinite. In this infinite point the hyperbolas tangential line would be 1. That infinite point would measured “from the light pulse” be equitemporal (follow the red light line back left downward) to *x*=*t*=0. Einstein is rumoured to have thought as a boy: “what would you see if you were sitting on a light pulse?”! The answer he later gave is clear from this: *nothing*. First because relative to light speed light stands still and could not reach your eyes. Second because you have mass and light speed is the unattainable upper limit of speed for hardware, that is, movers with a non zero mass. Oddly enough, even moving very near light speed relative to another mover you will measure the very same speed of light passing you: 299 792 458 $msec^{-1}$.

This should have prepared the reader for the details.

1. What is a hyperbola?

The general formula of the hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\pm 1$ (1) (why? Appendix A)



If $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = 1 (see *Graph 4*) you get a left-and-right hyperbola with -*a* and *a* as extremes, if $\frac{x^2}{a^2}-\frac{y^2}{b^2}$ = -1 you get an up-and-down hyperbola with *b* and -*b* as extremes. The graph shows curved line grids for line for many versions of $\frac{x^2}{a^2}-\frac{y^2}{b^2}$, that is for many values of a and b, keeping *a/b* the same so the two families of hyperbolas share the same two asymptotes *y*= ± (*b*/*a) x*. This is in preparation of what families of hyperbolas are going to look like in the space-time diagrams of relative geometry.

In all four directions the first three curves are omitted, which leaves the centre of the graph white. We could have done so and have filled this white space. But the lowest hyperbolas, where *a*=*b*=0 are drawn, in red: in the origin the four curves are degenerated to the red cross.

If necessary, check Appendix A on the basics of the hyperbola to make this fully clear. You will need it.

2. Space time diagram

The space time diagram hyperbolas have the forms $\frac{x^2}{a^2}-\frac{t^2}{b^2}=\pm 1$, where x is the distance between two movers (as measured by mover M with its orthogonal grid) and *t* is the clockposition at which this distance is measured (by M). Now the points in the graph are *events. Event* Z (see *Graph 5* below) happens at some distance *x* and some time *t*, read: “M measures event Z happening M-simultaneous with M ’s clockposition *t is b* and M -distancereading *x is f*, that is at (*x*,*t*)=(*b*,*f*)”. M ´ measures different values (*x´*,*t´*) for that same event Z. This (*x´*,*t´*) will appear only in later graphs.

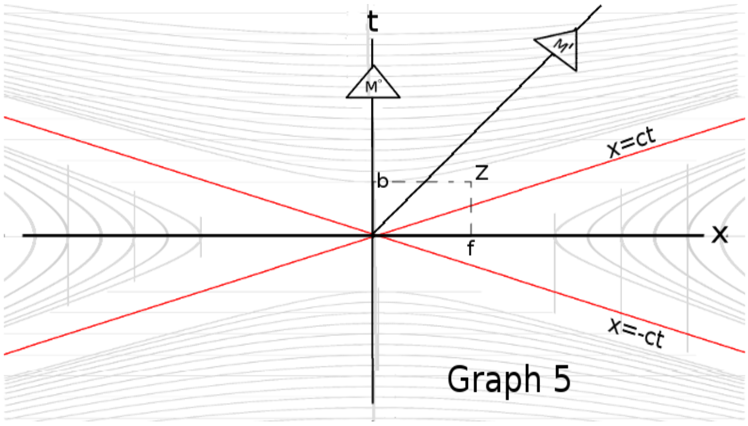
Thus, throughout we shall have to distinguish between time *and* clockposition *(one word, local orthography!). That is because same clockpositions of movers will be measured by those movers at different times. Similarly we distinguish between* distance *and* distancereading *(one word), for the identical distancereadings by different movers will not even give the distance to the same event at the moment where the two are together in the origin!*

A uniform mover M´ moving away from M with speed *v* experiences a series of home clockpositions t´. Those are called *events*. Each of these *events* enter the log of our zero mover M as (*x*,*t*). In (*x*,*t*), *t* is the clockposition of M (*not* the *t´* of M´ !) at which M clocks the event of M´ being at its M´ clockposition *t´*. M logs the movement of M´ as a linear series of events called *worldline* v:

$x=vt$ (2)

3. Normalizing the space-time diagram

If you take meters as x-unit and seconds as the t-unit, the graph will come out like this *Graph 5*:



Graph 5

The reason why the red lines are not orthogonal is that we measure in meters and seconds. For light speed is 299 792 458 $msec^{-1}$, roughly $3x10^8 msec^{-1}$. For light speed *c* (rounded) is $c = 3x10^8 x/t$, so the equations of the red lines read $x = \pm 1/3x10^8 t$. In fact, that would make the red lines in the graph unreadably close to horizontal. The graph as drawn already uses 10^8 m as the unit, so in *Graph 5*, $x = 3t$. That is, the red lines have slopes 1/3 and -1/3. We chose *c* as the fixed ratio $\pm b/a$ of the red asymptotes so the equations of the red lines are

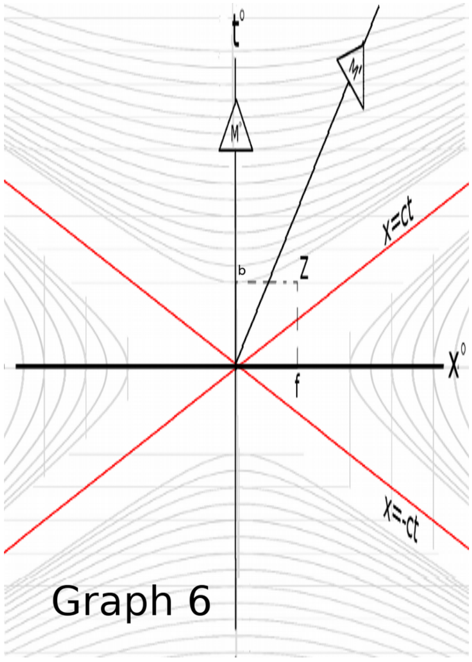
$x = \pm tb/a = \pm ct = \pm t/3$

That means if we have a value for *b*, the value for *a* is fixed by *c* as $c = \pm b/a$, thus $a = \pm b/c$ and substituting *c* for a in the formula for the hyperbola we get:

$$\frac{x^2}{(b/c)^2} - \frac{t^2}{(b)^2} = \pm 1 \quad (3)$$

Normalization of units means: setting the distance unit such that $c=x/t=1$ thus $x=t$. If we keep sticking to our seconds, what would the desired distance unit be? That means getting rid of that 3, precisely: that 2.99792458. Our distance unit should be $2.99792458x10^8$m, rounded $3x10^8$m.

We stick to seconds, so now, in the graph, the *graph length* (as you would measure with a ruler on a paper graph) of a second on the vertical axis should be equal to the graph-length of a lightsecond ($2.99792458x10^8$m) on the horizontal axis. Zooming the graph out, a year will graph-measure as of the same length as a lightyear. Zooming in: a nanosecond will have the same graph-length as a lightnanosecond (that would be 30 cm if your graph has scale 1:1).



Now the worldlines of the two light pulses going through (x,t)=(0,0) slope an elegant 45° that allows for easy overview of matters and efficient thinking, without any loss or distorting simplification. With c=1 hyperbola (3) simplifies to:

$$\frac{x^2}{b^2}-\frac{t^2}{b^2}=\pm 1 \quad \text{hence} \quad x^2-t^2=\pm b^2 \quad \text{hence} \quad x^2-t^2=\pm b^2 \quad (4)$$

By measuring distance in lightyears, lightseconds etc. we now have normalized the hyperbola, we have set a=b=c=1. In an ordinary image editor that means vertical scaling of *Graph 5* with 300% (precisely: 299.792458%) while keeping the horizontal width. I did so, to make *Graph 6*, *bluntly*, for that even stresses the point.

Thus the red lines are set 45°.

4. Defining the τ-parameter-family of t-hyperbolas

The formula $x^2-t^2=-b^2$ describes one hyperbola, that is one curve. We think of b as a fixed value. The family of curves for all such values shall be written as

$$x^2-t^2=-\tau^2 \quad \text{for} \quad \tau\in\Re \quad (6)$$

By taking fixed values τ=a, τ=b etc. you turn the generic formula $x^2-t^2=-\tau^2$ into a formula of a real hyperbola. Using expression (6) we can consider things generally: "*time frontier* τ". The up and down hyperbolas of Graph 6 are some of the infinite number of members of that *family.*

5. Deriving the algebraic equations of the hyperbola tangents.

Sticking to the positive side, hyperbola's of the family τ (6) have a minimum of τ. For $t\rightarrow\infty$, they have two asymptotes. One is $x=t$, the red line of the light pulse parting from the origin $(x,t)=(0,0)$ in forward, positive direction. The other is $x= - t$, the red line of the light pulse parting backward, in the negative direction.

We write the τ-hyperbola $x^2-t^2=-\tau^2$ as a function of t:

$$x=\pm\sqrt{t^2-\tau^2}=\pm(t^2-\tau^2)^{1/2}$$

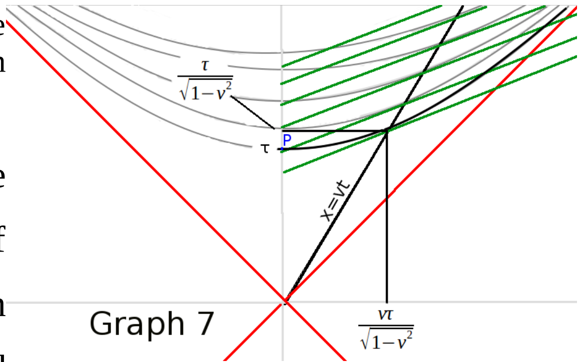
We calculate the derivative of the positive version (first quadrant) which the chain rule for derivation:

$$\frac{dx}{dt}=\frac{d(t^2-\tau^2)^{1/2}}{d(t^2-\tau^2)}\cdot\frac{d(t^2-\tau^2)}{dt}=\frac{1}{2}(t^2-\tau^2)^{-1/2}\cdot 2=\frac{t}{\sqrt{t^2-\tau^2}} \quad (5)$$

In the first quadrant near the point P: $(x,t)=(0,\tau)$ dx/dt approaches infinity, with means the tangent is getting horizontal. For $t\rightarrow\infty$, dx/dt tends to 1, the speed of light.

6. Important hyperbola property: tangents at the points of the τ-family of hyperbolas where, for some v, a worldline x=vt subsequently cuts through them, are all parallel with slope 1/v.

In *Graph 7*, six of these parallel tangent lines for the intersections of the τ-hyperbolas with $x=vt$ are shown (green).



Graph 7

Those tangent line slopes are all $1/v$ and hence all parallel. The proof is this: the points of intersection of $x^2-t^2=-\tau^2$ (6) and $x=vt$ (2) can be generally characterized by eliminating x and t in (6) and (2). Substitute x=vt in (6):

$$vt^2-t^2=-\tau^2 \quad \text{hence}$$

$$t=\pm\frac{\tau}{\sqrt{1-v^2}} \quad (7)$$

and

$$x=vt=\pm\frac{v\tau}{\sqrt{1-v^2}} \quad (8)$$

Taking the positive side, the value of $\frac{dx}{dt}=\frac{t}{\sqrt{t^2-\tau^2}}$ (5) at the point

$$(x,t)=\left(\frac{v\tau}{\sqrt{1-v^2}},\frac{\tau}{\sqrt{1-v^2}}\right) \quad (9)$$

is obtained by putting the t-value of (9) in (5):

$$\frac{dx}{dt}=\frac{\frac{\tau}{\sqrt{1-v^2}}}{\sqrt{\frac{\tau^2}{1-v^2}-\tau^2}}=\frac{1}{v} \quad (10)$$

Both numerator and denominator have factors *τ*, canceling each other out. The remaining expression, in v only, simplifies to 1/v.

Hence for different values of *τ* the tangent lines read:

$$vx=t-\psi \quad (11)$$

, for some ψ.

Choosing a value of *ψ*, however, means choosing a value for *τ*. We can get rid of *ψ* by expressing it in *τ*. We shall do that at the end of section 7 (formula (16>17)).

Where M´ ‘s worldline *x*=*vt* cuts through a hyperbola $x^2-t^2=-\tau^2$ for some *τ*, movers just slightly slower and faster will have clockposition *τ* when along the hyperbola, which at this point is approximated by the tangent line, and that tangent line, we now know, has a slope of 1/*v*. Locally that tangent line is the time frontier, not only when measuring near neighbours, but measuring everywhere:

For all values τ, all event-points on the entire tangent line of hyperbola $x^2-t^2=-\tau^2$ *, that is, the specific tangent line where x=vt cuts through this τ-hyperbola, are measured by M´ as happening at M´ ‘s clockposition τ.*

Fortunately, as we now know, for some value of v, that is, for some specific *x*=*vt* , the tangent lines through all hyperbolas $x^2-t^2=-\tau^2$ all have the same slope, so they form a neat series of parallel lines forming the time grid of M´ ‘s measurements. But movers with different speed *v* have different values for $\frac{dx}{dt}=\frac{1}{v}$ hence a differently sloping grid. In other words: they have a different linear selection of events (formula (11) for a different *v*) that are measured as simultaneous.

7. Using the hyperbola to calculate time dilation

In M ’s log, the event “M´ ’s clockposition is *a*” is timed on *b* (check *Graph 1* again). Strictly put: M measures the event S: “M´ ’s clockposition is *a*” as happening *simultaneous with* (“at the same time as”, “*equitemporal* to”) its home event P where its own clockposition is *b*. Thus, our zero-mover M measures the clockposition- event S (“M´ ’s clockposition is *a*”) as *later* than the event where his its own clockposition is a: event Q. The time difference (time dilation) is PQ in the graph. At this stage we only know the value of this time difference as *measured by M*.

In M ’s measurement, the event of point S: “M´ ’s clockposition is a” is simultaneous *to the event of point P: “M ’s - my - clockposition is b”, and* later *than Q: “M ’s - my - clockposition is a”*

Using the math of the hyperbola, *a* can now be expressed in *b* (the values derived below are put in *Graph 8*): M measures M´ 's clockposition *a* event point S as $(x_S, t_S) = (vb, b)$. Substituting (v*b*,*b*) in
$x^2 - t^2 = -a^2$ yields:
$(vb)^2 - b^2 = -a^2$ hence $a = b\sqrt{1-v^2}$
This means we can also express the *a*-hyperbola in *b* and drop *a* from the analysis altogether: $x^2 - t^2 = -b^2(1-v)$.

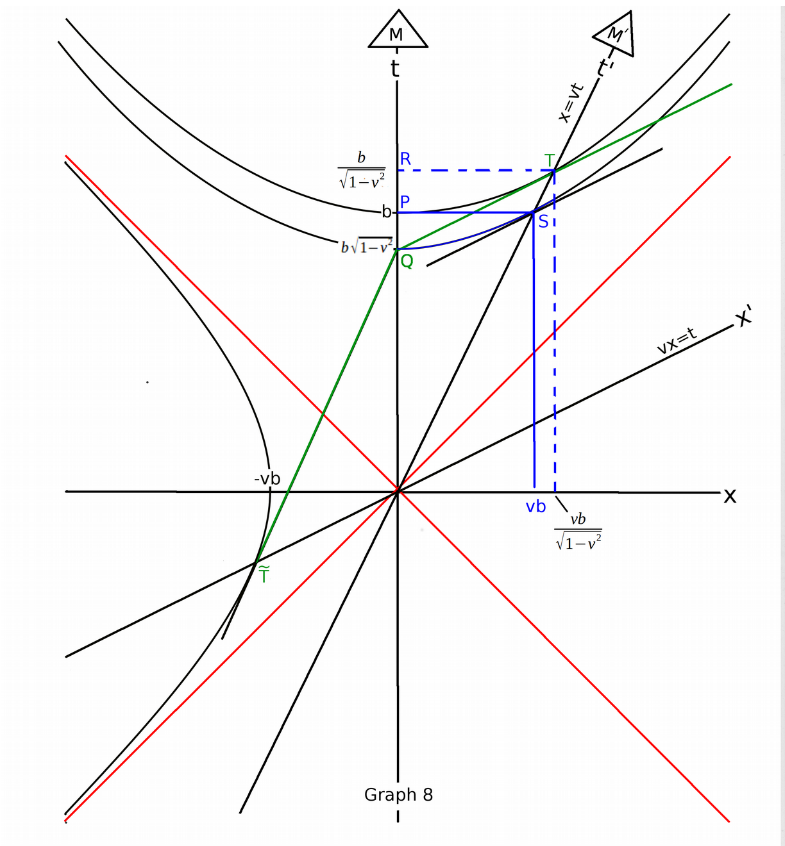
This allows us to express *time dilation*, for instance, the (clock-) position difference at M-time b between M´ 's clock (at S) and M's clock (at P). It is:

$$b - b\sqrt{1-v^2} = (1-\sqrt{1-v^2})b \quad (12)$$

Similarly, M's measurements of the event point T (M-equitemporal to point R in *Graph 8*) can be expressed in *b* by using *x*=*vt* (2) and $x^2 - t^2 = -b^2$ (4) solving (2) and (4) for (*x*,*t*), (we take the positive side only)

$(vt)^2 - t^2 = -b^2$ hence $t = \frac{b}{\sqrt{1-v^2}}$ It follows $x = vt = \frac{vb}{\sqrt{1-v^2}}$, hence

$$(x,t) = \left(\frac{vb}{\sqrt{1-v^2}}, \frac{b}{\sqrt{1-v^2}}\right) \quad (13)$$



Generalizing this from *b* to all values of *τ*, this also allows us to get rid of this *ψ* in the expression for the family of *τ*-tangent lines $vx=(t-\psi)$ (11), that is, the family of tangent lines of *τ*-hyperbola $x^2-t^2=-\tau^2$ (6) through one specific worldline *x*=*vt.* The generic description of the points of intersections (solving (*x*,*t*) in (6) and (11)) is analogous to (13)

$(x,t) = \left(\frac{v\tau}{\sqrt{1-v^2}} , \frac{\tau}{\sqrt{1-v^2}} \right)$ (14)

Put (14) in $vx=(t-\psi)$ (11):

$v\frac{v\tau}{\sqrt{1-v^2}}=\frac{\tau}{\sqrt{1-v^2}}-\psi$ (15)

solve for *ψ:*

$\psi=\tau\sqrt{1-v^2}$ (16)

So the tangent line $vx=t-\psi$ (11) becomes:

$vx=t-\tau\sqrt{1-v^2}$ (17)

This means now in (17) we can directly read from the equitemporal tangent line of a *τ* -hyperbola the clockposition *τ* of the *v*-mover when at hyperbola *τ*. Remember a *τ*-hyperbola itself connects the points where all movers (with all speeds -1<*v*<1) have that same clockposition *τ*. So every mover sees, at every single one of his own clockpositions *τ*

1. from $vx=t-\tau\sqrt{1-v^2}$ (17), along the straight line: all events *τ*-simultaneous *to himself* (*not* to movers with different speed!)

2. from $x^2-t^2=-\tau^2$ (6), along the hyperbolic curve with value *τ:* all events where other movers have the same clockposition *τ*

In section 11. we shall see that *τ* in (17) can be identified with *t´* of the Lorentz transformation for the time variable, so (17) is that transformation in disguise. The transformation will come by solving (17) for *τ*.

8. Using the hyperbola to calculate distance dilation

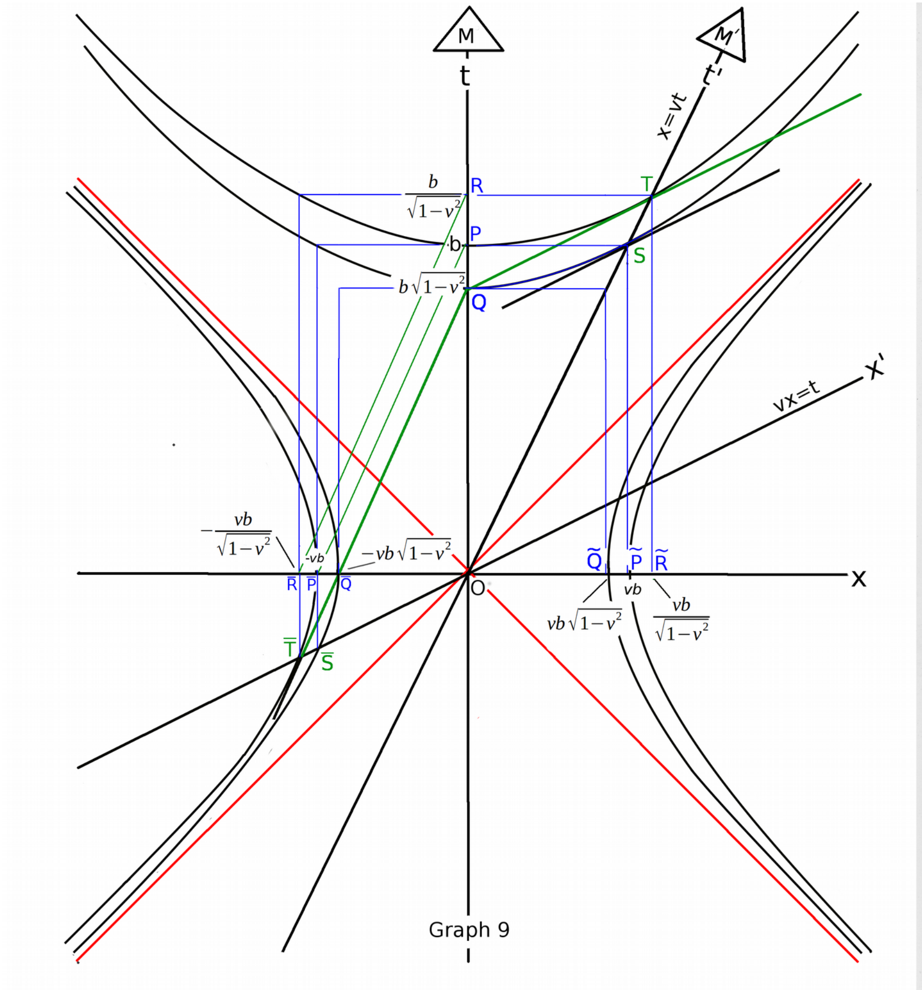
The procedure of calculating distance dilation is analogous to that of time dilation: It consist of adding to the (“up-down”) time hyperbolas *τ* the (“left-right”) distance *ξ*-hyperbolas (*Graph 9*):

$x^2-t^2=\xi^2$ (18)

For movers of all speeds -1<*v*<1, *ξ*-hyperbolas plot out curves of event-points that each have the distance *ξ*, each to one of those movers *when at at the origin.*

The following imaginary experiment was helpful to me: I imagine we launch a set of shiny balls, one for every mover (all speeds -1<*v*<1) to gauge, by launching a light pulse to bounce on it, in order to determine “his” ball’s distance *when the mover is at the origin*. And we want to launch them such that all movers will measure for “their own” ball the same distance *ξ* (we launch such that everybody identifies another ball, but all should measure the same distance *ξ*). Each mover identifies and distance-measures “his” ball. Then the space-time-position of all of these these balls at the moment they get flashed by “their” mover should be exactly on the hyperbole $x^2-t^2=\xi^2$ (18) as drawn in *Graph 9.*

The relation between the *τ*- and the *ξ*-hyperbolas is seen by first writing M’s x-values of M´ ‘s worldline *x*=*vt* corresponding to Q,P,R. Section 7 produced the algebraic expressions of the t-coordinates of Q,P,R. To obtain the corresponding algebraic expressions for $\tilde{Q},\tilde{P},\tilde{R}$ in *Graph 9* we multiply by *v* to get the expressions shown next to $\tilde{Q},\tilde{P},\tilde{R}$. We mirror them to the negative x-axis as well since M´ is obtaining the negative measured distances at $\overline{\mathrm{Q}}$,$\overline{\mathrm{P}}$ and $\overline{\mathrm{R}}$ when measuring te distance of M.



Graph 9

At the right side are the distance hyperbola's for positive ξ. The ones through $\tilde{Q}$ ($\xi= vb\sqrt{1-v^2}$) and $\tilde{P}$ (ξ=vb) and are drawn. At the left, the negative side, we drew the ones for $\bar{Q}$ ($\xi= -vb\sqrt{1-v^2}$) and $\bar{P}$ (ξ= -vb). At the time of event-point P, M measures M´ at distance *vb* of point $\tilde{P}$ d(PS)=d($O\tilde{P}$). M´ measures M at -*vb* over QT. At the origin the equidistance parallel of QT is O$\bar{T}$. $\bar{T}$ is not on the x axis but on M´ 's equitemporal line through O, and on the ξ-hyperbola with ξ=*vb*. M would read this distance as that equidistant (vertical for M) with point $\bar{R}$, that is: $-\frac{vb}{\sqrt{1-v^2}}$.

Distance dilation is analogous to time dilation. From M's point of view the analogon to time dilation (12) is *distance dilation* is $\tilde{Q}\tilde{P}$, from M´ 's point of few it is backward, negative, $\overline{QP}$

$$-vb\sqrt{1-v^2}-(-vb)=bv(1-\sqrt{1-v^2}) \quad (19)$$

Thus distance dilation always is *v* times the corresponding time dilation.
M´ 's equitemporal line from the origin can be written (*Graph 9*) :
v*x*=*t* *(20)*
Where (20) cuts through a ξ-hyperbola, the tangent of the hyperbola at that point is the locus of all events that have distance ξ from M´ at some point along M´ 's worldline *x*=*vt,* hence read (this is the ξ-analogon of (11)):

$$vt=x-\varphi \quad (21)$$

for some φ.

For φ=0 this is the worldline. On the worldline of M´, everything has distance 0 to M´. Choosing a value for φ means choosing a distance, a value for ξ . So we should be able to eliminate φ by expressing (21) in ξ and *v* only.

We want a generic description of any point where v*x*=*t* cuts through $x^2-t^2=\xi^2$ (18). Solving (*x*,*t*) from (18) and (21) yields (analogous to (14)) sticking to the positive side:

$$(x,t) = \left(\frac{\xi}{\sqrt{1-v^2}} \ , \ \frac{v\xi}{\sqrt{1-v^2}} \right) \quad (22)$$

Put the expressions for (*x*,*t*) of (22) in (21):

$$v\frac{v\xi}{\sqrt{1-v^2}}=\frac{\xi}{\sqrt{1-v^2}}-\varphi \quad (23)$$

Solve for φ (analogue of (16)):

$$\varphi=\xi\sqrt{1-v^2} \quad (24)$$

Substitute (24) in (21) to get analogue of (17):

$$x=vt+\xi\sqrt{1-v^2} \quad (25)$$

This means now in (25) we can directly read from the equidistance tangent line of a ξ-hyperbola the distancereading ξ of the *v*-mover of the object at hyperbola ξ. Remember a ξ-hyperbola itself connects the event-points to which some mover (with all speeds $-1<v<1$) has distancereading ξ from the origin. So every mover sees, at every single one of his own distancereading ξ

1. from $x=vt+\xi\sqrt{1-v^2}$ (25), along the straight line: all events at the distance ξ from himself from all points of his worldline

2. from $x^2-t^2=\xi^2$ (18), along the hyperbolic curve with value ξ*:* all events where other movers have the same distance reading ξ

Formula (25) is, like (17), a Lorentz transformation in disguise, this time the transformation of distance: solve for ξ (see section 12)

9. Symmetry proven

Symmetry holds when always if mover M measures some other mover M´ to be at M-distance-clockposition (*x*,*t*)=(*vb*,*b)*, then M´, in it own grid point notation (*x´*,*t´*) measures M at M´-distance-clockposition (*x´*,*t´*)=(-*vb*,*b).*

Why -vb (negative)? That is because when M´ is at a positive distance from M then M is at a negative distance from M´

Graph 9. The first steps of the proof are in section 7: P, where *t*=*b,* gives you S where M´ ‘s clockposition is $a=b\sqrt{1-v^2}$. Then S gives you (over the hyperbole) Q, where M’s clockposition is the same $a=b\sqrt{1-v^2}$

This remains to be proven:

1. *While measuring M at distance -vb,* M´ ‘s *clockposition is b.* That is: we have to prove that has the value such that tangent line $vx=t-\tau\sqrt{1-v^2}$ (17) goes through T, then: *τ*=*b.* To see this substitute (*x*,*t*)= $\left(\frac{vb}{\sqrt{1-v^2}},\frac{b}{\sqrt{1-v^2}}\right)$ (13) in (17)

$v\frac{vb}{\sqrt{1-v^2}}=\frac{b}{\sqrt{1-v^2}}-\tau\sqrt{1-v^2}$ and solve for *τ*. Speed *v* cancels out and *τ*=*b*, Q.E.D.

The second part of the claim to prove is:

2. *At M´-clockposition b, M´ measures M at a distance of -vb*. The simplest proof is this: When M´ moves from M with speed v, M moves from M´ with speed -v (minus, in the negative direction). Thus when M´ is at its M´-clockposition *b*, which we know from part 1 of this proof, it should measure the distance to M as *-vb*.

10. The famous Lorentz-transformation formula’s

The purpose of Lorentz transformation is this: our graphs up to now order events (points in the plane) following M ‘s measurements (*x*,*t*). M ‘s grid is nicely orthogonal, and M´ ‘s grid is skewed, though luckily rectilinear.

M´ ‘s grid is rectilinear only because throughout special relativity, theory is basically restricted to uniform speeds v: there are no gravitational or electromagnetic forces so there is no accelerating, decelerating or curving hardware in the space-time we consider.

This means that (*Graph 10*) every point measured by M as Z=(*x*,*t*) has another pair Z=(*x´*,*t´*), measured from M´. The two Lorentz equations form the algorithm that produces Z=(*x´*,*t´*) from Z=(*x*,t) for all Z in the plane. In their famous form the Lorentz equations read:

$$t'=\frac{t-\frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} \quad (26)$$

$$x'=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} \quad (27)$$

c is light speed, a universal constant. In par. 2 we set *c* to *c*≡1 , without any loss if information, by adjusting our distance unit *x* for that purpose. This means in our choice of units we can substitute *c* =1 in (26) and (27) to get:

$$t'=\frac{t-vx}{\sqrt{1-v^2}} \quad (28)$$

$$x'=\frac{x-vt}{\sqrt{1-v^2}} \quad (29)$$

So (28) and (29) are now the equations to derive, that is: to prove true for all event points Z=(*x*,*t*) which are, measured from M´, now written Z=(*x´*,*t´*).

11. Deriving the Lorentz time-transformation formula

Lorentz equation (28), the transformation of t to t´, can be obtained solving $vx = t - \tau\sqrt{1-v^2}$ (17) for τ:

$$\tau = \frac{t - vx}{\sqrt{1-v^2}} \quad (28)$$

Remember that along a *τ*-hyperbola all clocks read the same, hence *τ*=*t´*=*t*, and (17) is the equation of a tangent line of a *τ*-hyperbola at the point on the worldline of mover M´ exactly where he measures t´ for event Z.

12. Deriving the Lorentz distance-transformation formula

Lorentz equation (29) and *x´* are caught by considering the family of equidistance *ξ*-hyperbola (*Graph 10*) $x^2 - t^2 = \xi^2$ (18) and its associate equidistance tangents $x = vt + \xi\sqrt{1-v^2}$ (25) where a speed v mover's equitemporal line *xv*=*t* cutting through it. Solving (25) for *ξ:*

$$\xi = \frac{x - vt}{\sqrt{1-v^2}} \quad (29)$$

And indeed *ξ*=*x´*. For remember that along a *ξ*-hyperbola *ξ*=*x*=*x´*, and $x = vt + \xi\sqrt{1-v^2}$ (25) is the equation of a tangent line of a *ξ*-hyperbola at the point on *xv*=*t* where mover M´ measures exactly *x´* for event Z.

In our example, both *x* and *x´* are postive since to both movers, Z is forward. But Z can be between them or at the rear of both and signs for distance measurements will change.

Realizing *ξ*=*x´* took me some more time to realize than realizing that *t´*=*τ*. That, I think, was because physically, the *τ*-*ξ* analogy is *not* perfect: an event involving some M can read: "M's clockposition is *a*". But there is no analogue of distance "M's distance reading is *vb*", for *distancereading,* unlike *clockposition, is relative to another mover M´, hence cannot independently characterize an* event. Clockposition readings are "absolute" in the sense of being independent of any other mover, and thus do really on their own characterize events.

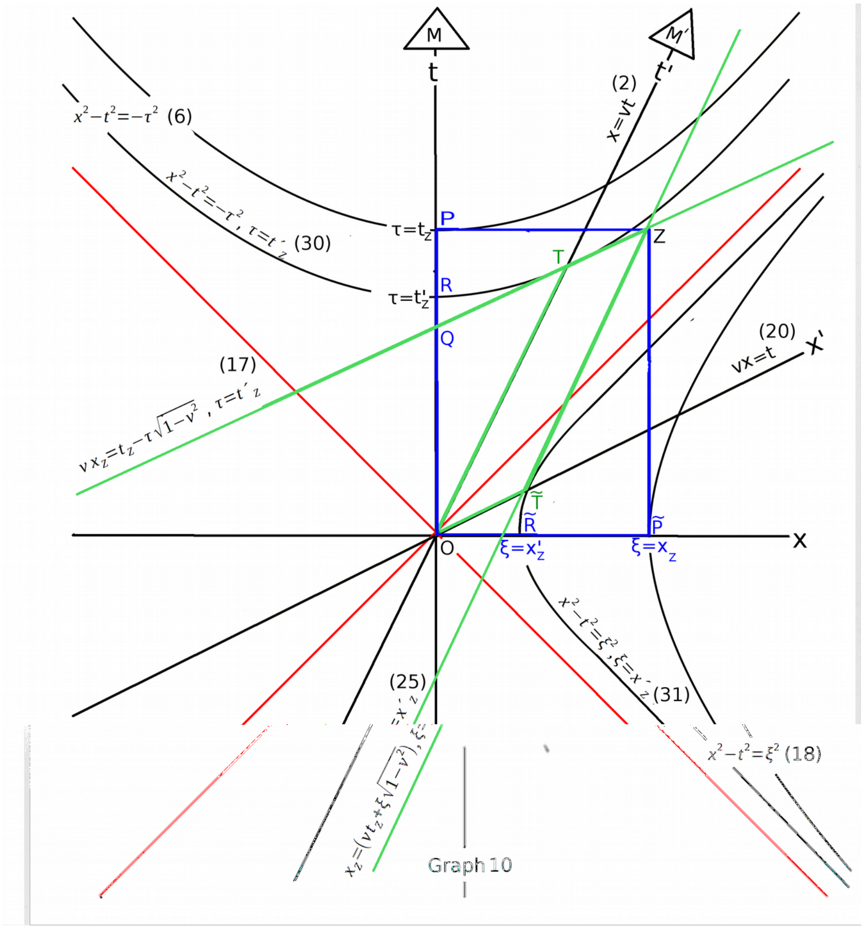
But there is another sense in which, on the contrary, it is distance which is the "absolute" variable of the two: setting time to zero, as we did in the origin, is an arbitrary decision. You can sync clocks at another pair of events ((x,t), (x´,t´)), *while setting distance to zero at the origin only makes sense ony if M and M´* really *have the same position,* really *are at the same place.* And being at the same place the same moment is an *absolute* measurement, for when movers are together at one single position, all simultaneity divergence issues have vanished.

Despite these differences in the nature of time and distance, in the *mathematical* aspect the analogy is perfect: the line $O\widetilde{P}$ in *Graph 10* measures x_Z, Z's M-equitemporal distance to M (at M's time t_Z), and $O\widetilde{T}$ measures, in M´ 's grid, $x´_Z$, Z's M´-equitemporal distance to M´ at M´ 's time $t´_Z$. $O\widetilde{R}$ measures the same value $x´_Z$, but in M's grid. Movers with other speeds will have *ξ*-tangents to other *ξ*-hyperbolas when measuring that same Z, that is, measure yet another value for their distance *x* to Z. If we consider one equidistance hyperbola for one value of *ξ*, then for every mover 0<*v*<1 this represents a different event. Some may be helped here by putting it thus: the point where the isotime tangent line though the origin, v´´x=t, of some third mover M´´ cuts through a *ξ*-hyperbola on which M measures its distance to Z as *ξ* (thus $\xi = x_Z$), tells you where Z *should have been* to be measured at a distance *ξ* by M´´: it's a different event, a point different from Z in space-time.

Hence: though the formulas for ξ and τ are perfectly analogous, the physical interpretations of x and ξ differ from those of t and τ.

Summary: Lorentz from scratch on one page (looking at *Graph 10*)

We refer to the formula's in brackets, numbered as in *Graph 10*.



Measured in the orthogonal grid of mover M, the movement of mover M´ with uniform speed v is a linear sequence of events (2). In every subsequent event on (2) M´ experiences a next home clockposition $t´$. At T, for reasons that will become clear below this clockposition is $t´_Z$. For now it only matters that T has *some fixed value* $t´_Z$ *of* $t´$. Hyperbola (30) represents the locus of all events where all movers with all speeds $-1<v<1$ experience a home clockposition of this specific value $t´_Z$. So (30) is the *time frontier* $t´_Z$: above (30) , every mover's clockposition is *later* than $t´_Z$, below (30), every mover's clockposition is *earlier* than $t´_Z$. Locally at T, time frontier (30) slopes up conforming T's tangent (17) the slope of which is determined by the speed v of M´. Hence (17) represents simultaneity as it is experienced by M´ at T, in other words, all event-points in space-time *simultaneous* to event T in M´ s measurement grid are on (17). Off (17), at R, it is M who experiences that same specific clockposition $t´_Z$. But at R the tangent line of (30) is horizontal.

It follows from the mathematical properties of the hyperbole that all tangent lines of all time frontier hyperboles of type (6) for any $\tau \in \Re$ through worldline (2) of M´ are parallel and of form (17) for some τ. In other words: when you change the value of τ in (17), you get another line parallel to the (17)-version drawn in *Graph 10.* And this line will be a tangent of the hyperbole exactly at its intersection with (2). M has similar tangent lines, but these are all horizontal, like the one through P and the one through R.

Now consider the event Z. Z is measured by M as (t_Z,x_Z) and by M´ as ($t´_Z$,$x´_Z$). we can directly mark the coordinates (t_Z,x_Z) on the orthogonal axes of M. For Lorentz transformation of (t_Z,x_Z) into ($t´_Z$,$x´_Z$) we need to express $t´_Z$ in t_Z and $x´_Z$ in x_Z , that is: we need to find out where to mark $t´_Z$ and $x´_Z$ *on the orthogonal axes as well*. Now I have to justify that I put those marks exactly where I did so in *Graph 10*, that is: at points R and $\tilde{R}$

For $t´_Z$ my justification is as follows: how does M´ measure the distance it had to Z when Z occurred? M´ must check its log for the exact time at which Z occurred, and check its distance from Z at "that moment". At "that moment"! That is: in its log, M´ has to find back the position where it was "when event Z happened", that is its position when equitemporal to event Z. We call it T. The tangent of the τ-hyperbole through T is that of $\tau= t´_Z$ (30). So now $t´_Z$ is no longer to be taken, as we did above, as just an arbitrary value of τ. This $t´_Z$ now is, in M´ 's measurement, *M´ 's clockposition at which event Z occurred*. The event where the other mover, M, had that same clockposition is found by following from T the hyperbole (30) to M's vertical t-axis, where you arrive at point R. So at R we can mark the value of $t´_Z$ on M's vertical t-axis (and so we did in *Graph 10*).

The algorithm for this transformation derives from (17), which actually is an implicit form of the Lorentz transformation of time: the algebraic formula for the connection from P to R (over Z and T) is found by substituting setting $\tau=t´_Z$ in (17) and solving for $t´_Z$ you get $t´=\frac{t-vx}{\sqrt{1-v^2}}$ (28). I dropped the Z index as well, to generalize to any Z in the plane, that is, any Z in space-time.

Through analogous mathematical operations we get the transformation of distance (29): For $x´_Z$ we do it as follows: the distance TZ in M´ 's measurement is equal to the distance $O\tilde{T}$ (the points Z, T, $O,\tilde{T}$ form a parallelogram). $\tilde{T}$ is on the distance frontier $\xi= x´_Z$ (31). M measures the same distance on the same distance frontier but since its measurements are set orthogonal, it does so in in horizontal direction, that is, at $\tilde{R}$. That is where you have to mark the value of $x´_Z$ on M's horizontal *x*-axis (as done in *Graph 10*).

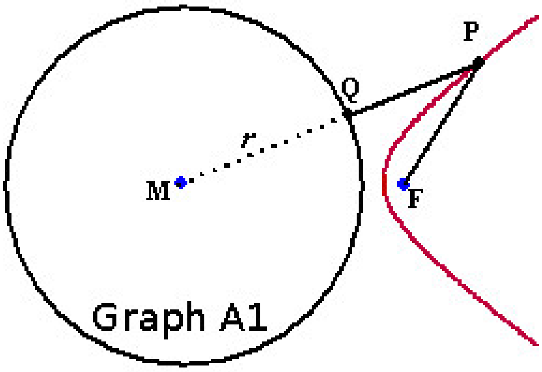
The algorithm for this transformation derives from (25), which actually is an implicit form of the Lorentz transformation of distance: the algebraic formula for the connection from $\tilde{P}$ to $\tilde{R}$ (over Z and $\tilde{T}$) is found by substituting, setting $\xi=x´_Z$, in (25) and solving for $x´_Z$ you get $x´=\frac{x-vt}{\sqrt{1-v^2}}$ (29). Again, I dropped the Z index as well, to generalize to any Z in the plane, that is, any Z in space-time.

Appendix A: the hyperbola at high school

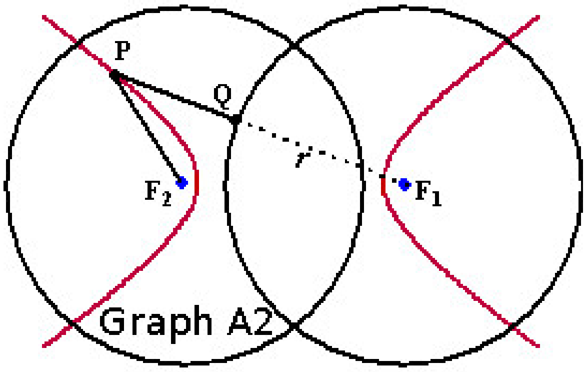
translation of
http://www.hhofstede.nl/modules/hyperbool.htm and http://www.hhofstede.nl/modules/hyperboolasymp.htm
from http://www.hhofstede.nl/,
the unparalleled Dutch reference website, said of highschool mathematics but in fact much more
by H. Hofstede, Warffum, Groningen.

The hyperbola can be defined in many ways but the one most efficient to begin with is by using the concept of the *equidistance line* (or "conflict" line).

It is the line of equidistance to a circle and a point outside that circle, the red line in *Graph A1*: PQ=PF.
But because PQ=MP-*r*, it follows that MP-*r*=PF. Hence MP-PF=*r*.
That is why circle centre M is also called a *focus point* F.



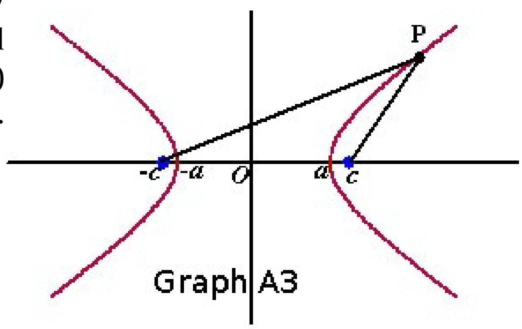
So now (*Graph A2*) we write F_2 instead of M and F_1 instead of F. We wrote MP-PF=*r*, now we write $F_2P-PF_1=r$. Adding the left-right mirror yields a second branch where also $F_1P-F_2P=r$. These two can be written as $|F_1P-F_2P|=r$
Graph A2 already makes clear that the distance between the extremes equal r.



So for a hyperbole: | *d*(P,F1)-*d*(P,F2)| = *r*

(*d*: point distance)

We construe an algebraic formula for the hyperbola by choosing the middle of the figure as the origin of the grid (*Graph A3*). We denote the focus point coordinates by (*c*,0) and (-*c*,0), and the extreme points by (-*a*,0) and (*a*,0). Hence *r*=2*a*.



At P: (x,y) we apply Pythagoras twice:

$$d(P,F_1)=\sqrt{((x-c)^2+y^2)}$$

$$d(P,F_2)=\sqrt{((x+c)^2+y^2)}$$

Hence:

$$\sqrt{((x-c)^2+y^2)}-\sqrt{((x+c)^2+y^2)}=2a$$

which substituting $b^2=c^2- a^2$ simplifies to:

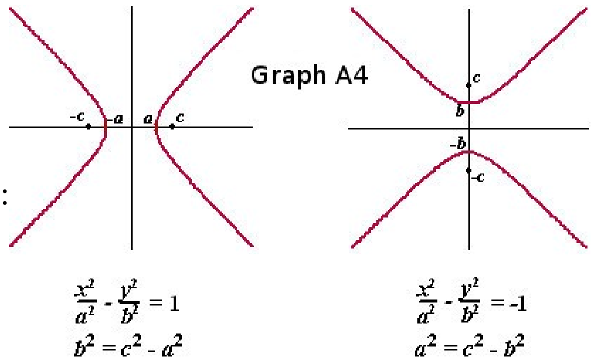
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

If P is chosen on the left branch the result will be the same.

If you interchange $\frac{x^2}{a^2}$ and $\frac{y^2}{b^2}$ to get

$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ hence $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$

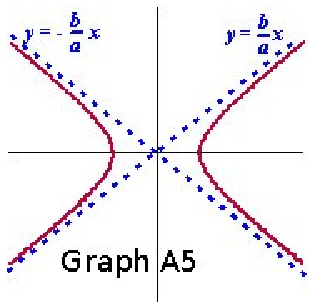
you get a hyperbola as on the right side of *Graph A4*: up and down with extremes b and -b.



The hyperbola has two linear asymptotes: for x and y $\rightarrow\infty$ consider:

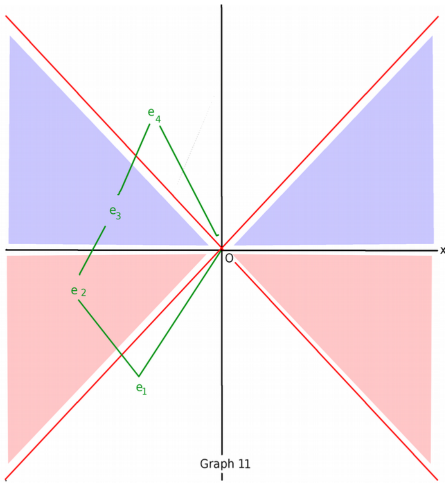
$\frac{y^2}{b^2}=\frac{x^2}{a^2}\pm 1$ hence $y=\pm\frac{b}{a}\sqrt{x^2\pm a^2}$

for x $\rightarrow\infty$, $\sqrt{x^2\pm a^2}=\sqrt{x^2}=x$ hence $y=\pm\frac{b}{a}x$



Appendix B: what's more to see in the space-time diagram.

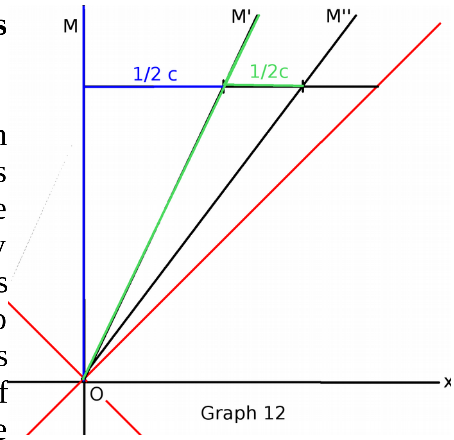
Causality



Suppose the event in the origin (0,0) is a crime scene. Anybody who previously was at an event in the pink area has a perfect alibi. Anybody who afterwards was at an event in the blue area has a perfect alibi as well.

This does not imply those areas are perfectly irrelevant to a detective: crime scene O can be in a partial causal ordering of events going through the coloured areas, and who knows e_4 is the real thing a ring of criminals, starting at e_1, wanted to cause. And the same holds for causes and effects in spacetime generally.

Proper concatenation of speeds not what humankind always presumed it was



The geometry of special relativity shows that though on earth you can catch a rabbit from a running horse by adding the speeds of both in aiming your spear, or arrive at the same time at the beach with a fast and a slow car, this really is only a handy tangential approximation at low speeds: in *Graph 11* M´ moves relative to M with half the light speed. And M´´ moves relative to M´ with half the light speed. That does *not* mean that M´´ moves relative to M with twice half = the full light speed, but only 4/5 of light speed (see G*raph 11*). The formula follows from doing the Lorentz transformation between M´ and M and then transforming the transformed (x´,t´) again to (x´´,t´´) and defining v_1=v(M,M´), v_2=v(M´,M´´) and v_3=v(M,M´´). That nested transformation simplifies to:

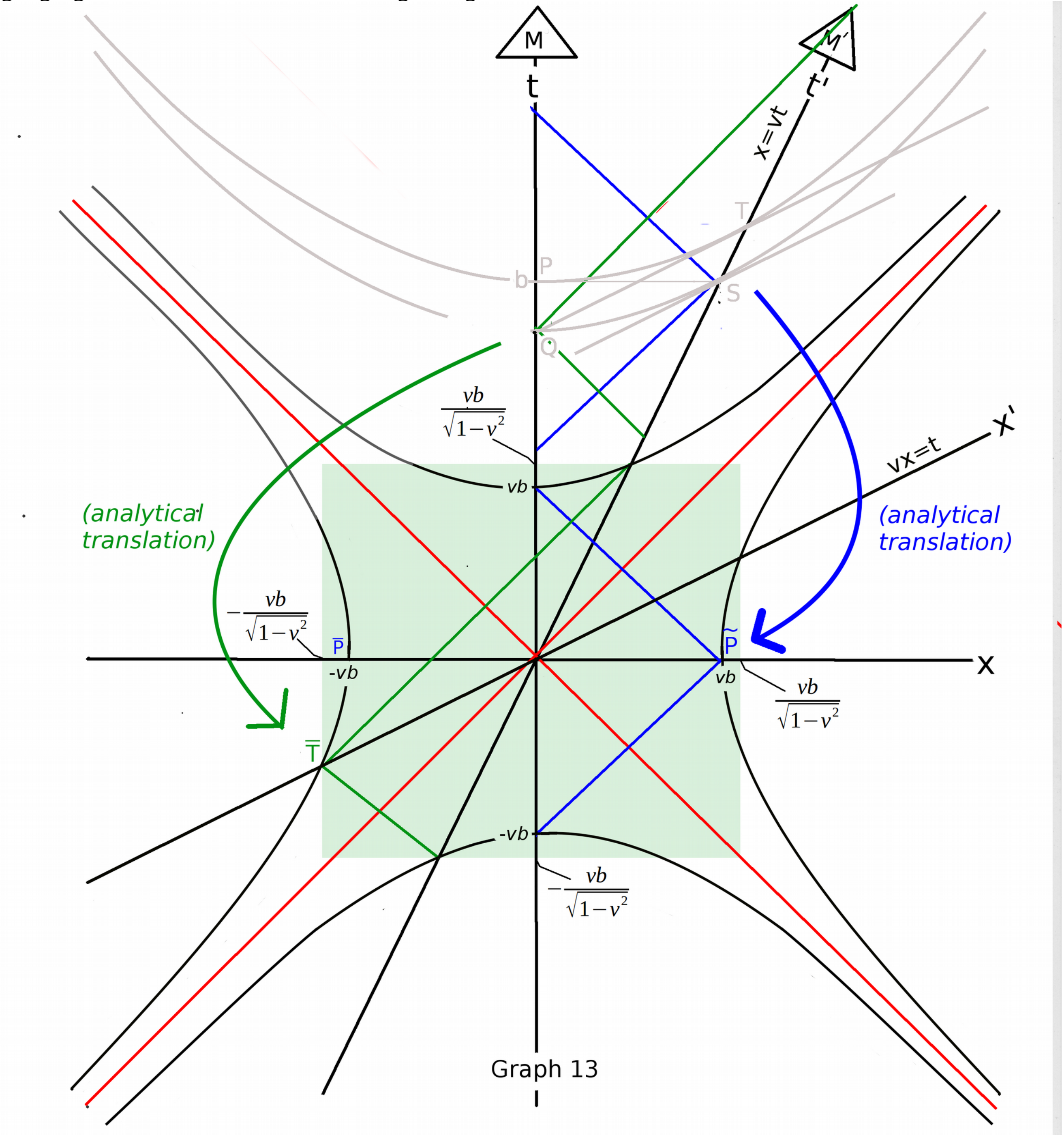
$$v_3 = \frac{v_1 + v_2}{1 + v_1 v_2}$$

In the example v_3 = 4/5. This shows again that light speed c=1 is unattainable for moving hardware: how ever close both v_1 and v_2 come to light speed =1, v_3 will never reach =1.

How movers measure distance

We dealt with movers' distance measurements without bothering how such distances actually could be measured. When gauging is done by creating a light pulse reflection on the measured object, a mover has to shoot the pulse before "the moment" he wants to measure: the pulse needs time to reach the object. Moreover, you calculate the distance your pulse traveled to and fro by timing it and using the universal constant of light speed, so you will only know the result after "the moment", since the pulse needs time to get back. With "the moment" we mean *the moment where the object is while simultaneous with the home clockposition at which the mover intends to measure the distance*.

This shows that simultaneity in itself not independently observable: a mover needs to measure whether some away-event is simultaneous with with some home event, thus needs relativity theory to understand what he is doing. In other words: to believe in your simultaneity tangents you have to believe in relativity theory first. That is why text-books that start shooting pulses around to explain relativity, take a logically impossible explanatory direction. We try better by addressing the issue of gauging here, at the end, not at the beginning.



We analyse the measurements involved in our graph-illustrations thus far. High in Graph 13 you see the situation we used, but greyed out. There now is blue line going right-up, making a rectangular turn at S then going left up. That is M ‘s gauge of M´ ‘s position at P. We took it to be *vb* as in the example we used throughout. *Graph 13* makes clear that the gauging pulse is launched before P and that while gauging for the distance from P, M does nothing at event P itself but waiting for the light pulse to return! M´ ‘s gauge of M (green), with a pulse bouncing on M at Q, is analogous.

Without changing any conclusion from this analysis we can simplify graph reading by moving both gauging procedures down to were M's orthogonally presented gauge targets something at an M-distance *vb* from the origin, that is, at t=t´=0. Think of, instead of gauging each other, M gauges a shiny ball we put at distance *vb* from M. And we give M´ another ball at -vb. It follows that in the space-time graph we should hang M's gauging target ball on the x-axis, exactly vertical under S in the space-time graph, to make the ball's distance to M equal to *vb.*
M should shoot his gauging pulse (the blue half-square through $\widetilde{P}$) in negative time t=-vb to reach his ball at t=0, and will get the pulse back at t=vb since his light pulse line (drawn with M's own grid orthogonal) makes a neat half-square.

M will launch the pulse at: $(x,t)= (0,-vb)$
The pulse will bounce at: $(x,t)= (vb,0)$
M will get the pulse back at: $(0,vb)=$
Total time for the pulse to and fro: 2vb
Total distance made by the pulse to and fro (for we set c=1) : 2vb
The distance of M´ as measured by M thus is half the distance his pulse travelled: *vb*

But measured in M, M´ 's pulse (the green line throught $\overline{\text{T}}$) does *not* make a half square for when M´ launches the pulse, the point on M´ worldline from where M´ shoots is *left* of t-axis, and where he gets it back after he moved to the *right* of it. So now where should we hang a ball for M´ to gauge it -*vb* when M´-simultaneous to M´ when M' is in the orgin?

Written in M's grid, (not M´ 's own grid!), M´ will send the pulse from his worldline $x=vt$ at t= $-\frac{vb}{\sqrt{1-v^2}}$. In graph 13 all edges of the the transparant square green surface are at distance $\frac{vb}{\sqrt{1-v^2}}$ from the origin. M´ will get it back on his worldline $x=vt$ at t = $\frac{vb}{\sqrt{1-v^2}}$ in the event:
$(x,t)= \left(\frac{v^2 b}{\sqrt{1-v^2}} \;,\; \frac{vb}{\sqrt{1-v^2}} \right)$

(this *x* is not shown)

The total time between launch and return of the pulse is

$$\frac{vb}{\sqrt{1-v^2}} - -\frac{vb}{\sqrt{1-v^2}} = 2\frac{vb}{\sqrt{1-v^2}}$$

Remember $\frac{1}{\sqrt{1-v^2}}>1$. So it took M´ 's pulse more than *vb* time to go from and return to M´, measured in M's grid like we do in *Graph 13*. That is because in M's grid, M´, while gauging, moved away from his gauging target, and his light pulse has to catch up with M´ which needs extra time. If you substract M´ 's increase of distance from his gauging target while gauging, exactly *vb* remains, to be precise: -*vb*, for M´ is gauging in the negative direction.

Another way to see that -*vb* should come out as the home distance-reading by M´ when M´ target is at $\overline{\text{T}}$ is this: shift to M´ 's own grid, that is, make M´ 's grid orthogonal instead of M's, like we did in *Graph 3*, by means of a horizontal flip (*Graph 14*). Now, after the horizontal flip, the vertical axis is M´ 's worldline instead of M's and the half-square of blue pulse lines will now be M´ 's. Now it is no longer M´ ‘s, but M ‘s measurement that should be corrected for the $1/\sqrt{1-v^2}$. You see the very same gauging process log. You only Lorentz-tranformed the graph.