Lorentz Transformation Made Easy

How light shapes the geometry of relative space-time

Comprehensive, only basic math, 28 pages, 20 hours proper studying

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How light shapes the geometry of relative space-time

Comprehensive, only basic math, 25 pages, 20 hours proper studying

Most relativity teaching literature is written by very talented teachers for very talented students. The teacher skips a lot of steps and students need only half a word and do not even notice.

This short explanation of the geometrics of special relativity is in words and images that should keep many more readers on board. It is the logbook kept and refined by a curious ignorant outsider who needed every single step while struggling hard to grasp the subject and persevered in searching and brooding until he got them all. The primary purpose of this text is to tell how I now explain Lorentz transformation to myself way easier than in any text I have seen. To me, it lost all "counter-intuitive" haze.

At the end a reader equipped with highschool math is supposed to be able to Lorentz-transform coordinates while exactly understanding what he does.

I am grateful to Profs Gerard ‘t Hooft, David Atkinson and Fred Udo to occasionally help a complete outsider finding his way. I entered some other scientific fields as a stranger but never was received so kindly.

Bert Hamminga (http://asb4.com/aboutme.html)

Internet version: http://asb4.com/relativity/Lorentz Transformation.pdf
Separate graph sheets: http://asb4.com/relativity/Lorentz Transformation Graphs.pdf
Project blog: http://asb4.com/relativity/index.html

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1. The problem

The problem to which relativity theory is the solution is the discovery in the late 18th century, of the absolute speed of light: from whatever position, moving relatively to other positions or not, light speed *c* is measured as 299 792 458 m/s. I was told so at high school in the 1960's, just memorized it without asking myself whether I should worry about all I had learned on positions, including sequences of clock positions, distances, dimensions of objects and and speeds. As far as I remember it failed to push me in vertigo. Wrongly. It should have done so. It upsets everything. The problem is this:

Think of a light flash passing two pieces of hardware in space:

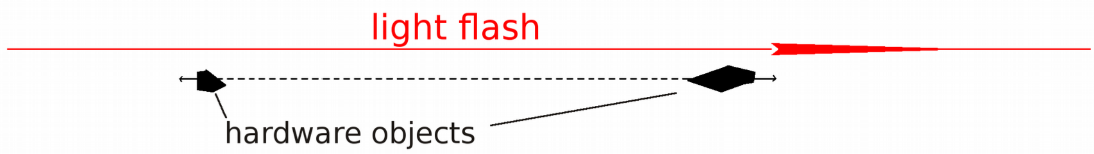

Fig. 1

From both hardware objects light-speed is measured to be *c*. This seems nothing to worry about, even natural, if you consider these two one by one. How could light speed be different to them? It starts to get puzzling once you consider a case in which the two objects are moving relative to each other. Think of them moving away from each other at considerable speed relative to light, say half of light speed, along the line of the light flash. Now customary non-relative habits of physical reasoning tell you that a light flash from the left should pass the left object faster than the right one: "if you would sit on the light flash", everything seems to tell you, "the left object should pass faster than the right one".[1])

More precisely, these wrong habits suggest that if the flash passes the left object with speed *c*, and the right object moves away from the left object in the same direction as the light pulse does, with speed *v* (*v*<*c*), the flash should pass the right object with speed *c*-*v*. The habit is forgivable for a biological species like the human hunter, running, as is his method of survival, after a prey. From the perspective of speeds in the order of light, both creep invisibly slowly over the earth's crust: if the light flash were a buck moving away from a sitting spectator with speed c while it outruns a hunter pursuing it (the right object) with speed *v* (relative to the spectator), the hunter, would he have had time for some contemplation would have "inferred" that the buck moves away from him with relative speed *c*-*v*. But if you travel at a decent portion of light speed the error gets measurable and nasty. Even in the trio buck–hunter–spectator the addition of speeds results in an error (far too small for the hunter to notice). Our brains, genetically hunter–brains really, do not correct for it since never in the history of our species, nor of any other on earth, a higher survival chance has

1) When young, Einstein famously asked himself: what would you see if you sat on a light-flash? His answer was: nothing

been associated with brains endowed with such a sophisticated neuro-gadget. The error proved evolutionary acceptable for biological species on earth[2]). In human society it used to be the same, at least in the states of knowledge and technical performance until 150 years ago. We shall precisely evaluate it.

The solution to the problem of “the missing *v*” was found by considering precisely how meters and the seconds are measured. The amazing thing is that the way we do it, did it and always thought about it, implies that for the two objects of Fig. 1 they have different outcomes: they are not the same meters and seconds: measured from one object, what is measured to be one meter by the other is not a meter, and the same with the seconds. And this distance-unit and time-unit transformation exactly accounts for the *v*-factor in our customary erroneous non-relative way of reasoning. Their speed relative to one another does not make the two objects measure light speed differently, but to measure each other’s meters and seconds differently. Think of comparing England and Egypt and finding out that the Egyptian and British pounds do not have the same value, that the British and the Egyptians both tend to calculate and memorize prices in pounds of their own country.

Suppose you are on the left object of Fig. 1 and I am on the right one. The light pulse passes me while I move away from you in the same direction as the light flash. Yet, I log the speed of the flash as the very same 299 792 458 m/s as you do, and not less, as your habitual thinking makes you expect. Now you could consider the possibility that my meters are shorter, my seconds longer, or a combination of both. If my meter-rod would be shorter than yours – in your perception - and my clock runs slower than yours – in your perception - I would log a higher light speed (m/sec) then the *c* minus *v* you expect me to measure. With smartly chosen rod length and/or clock-speed differences I could even measure exactly the same light speed *c* as you did. As in fact I do.

If you look for such a type of a solution you should be ready to measure space and time in two ways: from the perspective of the left object, and from the perspective of the right object. We have two standards. One could add more objects (and with it, standards) to the example but that is of no help to solve the problem. What we need to know is how the two standards (two pairs of meters-seconds-units), that we now have, relate to each other. The problem is to find what the transformation of the meters and seconds of one object to those of the other depends on, and what mathematical equations describe the transformation.

Doing that we give up the idea that space has a geometry (a set of coordinates, grids, for position and time) *of itself*. If you have two movers in space, each has its own geometry, so there are two. There is no third as long as there is no third mover, which is of no help to solve the problem. If the coordinates of the two geometries would relate according to a fixed formula then in communicating

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2) During evolution, other, more complicated corrections got built in the brains of biological species: chimps throw stones fast and far, with amazing precision, over elliptical orbits. Humans learned to correct for light diffraction when spear-fishing. But the archer fish is in a different league: while spitting water drops from below the water surface at insects sitting on branches above the water, it corrects for light diffraction at the water/air border, air resistance and even the elliptic curving of the water drop orbit under gravitation, that the drop is not affected by its own medium that it shares with the fish. From the viewpoint of physics this is in the league of a Mars landing.

positions, time and speeds the two movers can transform any plot-data received from the other into their own coordinates.

These are all pieces of the puzzle. The solution is called Lorentz transformation. It is a matter of pure mathematics. The math was derived in the mid 1880's by Lorentz for movement in electrodynamics (light experiments). There are two equations: one for the meters and one for the seconds. In 1905, Poincaré coined the term *Lorentz transformation* for them. In the same year Einstein put them to general use by dropping the idea of a medium ("aether") transmitting light (which had been coined in analogy to sound-transmitting air) and applying them to all movement generally. What followed were dramatic simplifications that now make it fairly simple to understand special relativity with the help of highschool mathematics.

2. How to graphically represent moving light and moving objects

Consider an object C in space. It makes no sense to ask whether it moves or not. That would require another object relative to which it would be moving. It is in uniform motion: neither accelerating nor rotating, if it had a space somewhere inside with a loose object in it, as the gravity sensor of your smart phone, this object would not be pressed against any of the walls and float freely. A phone screen would never auto-rotate.
From C, two light flashes are launched along one single, straight line, in opposite directions. One we call the rightward flash, the other the leftward. On C, a clock is set zero at launch.
In a graph, we plot clock-readings by C vertical up and distance horizontal.

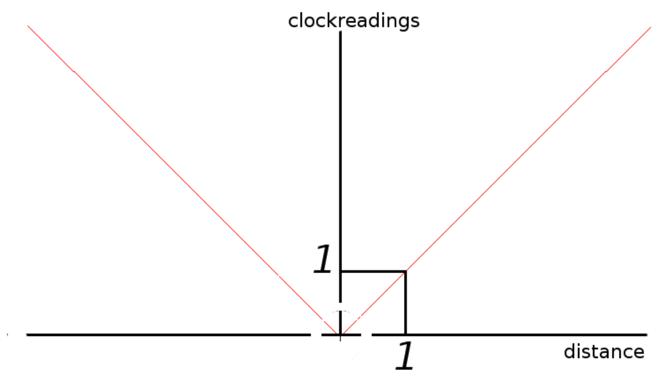

Fig. 2

It is called a *Minkovski* or *space-time* diagram. We do not consider the space off the line. Our space of analysis is this one line. Generalization to a real three-dimensional space would not change the basic outcome and this limited set-up neatly focuses the brain on the steps to make. In the graph, plotting time vertical the time-path of the rightward light-flash goes to the right up, the leftward flash to the left up. How steep? That depends on the units of the graph axes. If you choose as the time unit the second, and as the distance unit the light second (= 299 792 458 m), the light flash lines will slope at exactly 45°. Speed of light, in those units, equals 1 (one) by definition. You can always choose units in such proportions. Analyzing cosmic distances you might for instance take years and light-years, with the same result. Applying this procedure is called *normalization*. Setting c=1 obscures nothing and simplifies formulas.

What is a “clock”? Any regular process that has countable events can be used a clock. The sun clocks years and days, sand glasses count turns, pendulum clocks count swings etc. The relative precision of different types of clocks can be measured in comparative tests.

We focus on things moving on one single line (uniform motion), and the graphs keep track of that movement by plotting time vertically, so speeds (leftward and rightward on that single line) can be read as the angles between the graph-lines: low speed makes a small angle to the vertical axis, high speed a bigger one, the speed of light has a 45° angle to the vertical axis.

We add two movers, B and D. D moves rightward and B leftward relative to C, with equal speed, on the same line as the flashes, so the distance of B and D to their middle C will, while growing, remain the same. At departure B, and D set their clocks to zero, as C did. To measure the distances of B and D, C at some moment (event e_l in Fig. 3) launches light flashes in both directions at exactly the same time. Those flashes are mirrored by B and C to return to C. Light speed is fixed, hence the time elapsing between launch and return of the light flashes will tell C at what distance B and D were at the moment the flash reached their mirrors and bounced back. If C is properly in the middle, flashes launched simultaneously leftward to B and rightward to D should return at the same time.

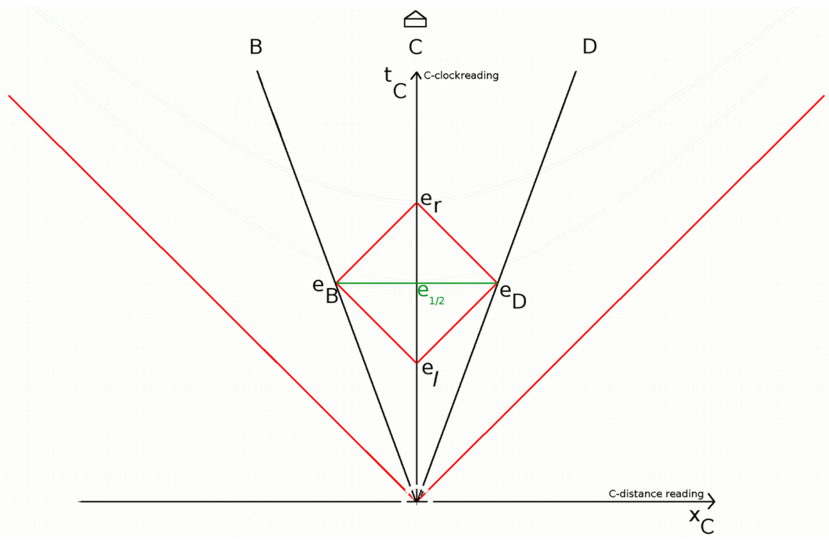

Fig. 3

Fig. 3 shows the flash procedure that shall be referred to in the steps to come. The black straight lines show how A, B and C move away from each other at constant speed and are called *world lines*. Using x_C for C's distance-readings and t_C for C's clock-readings, all points in the graph have coordinates (x_C,t_C) and plot what are called *events in C's* measurement. Where a mover's world line cuts through an event, this mover can do a clock-reading of it. Strictly, every reading of a position of the clock of a mover is itself an *event* on the mover's world line. Coordinates *are* no events; instead, events *have* coordinates. Those coordinates will turn out to have different values if measured from the perspective of different movers B, C, D, … Comparing those perspectives is what Lorentz transformation is about.

Fig. 3 shows five events marked *e*:
Event e_l: C launches two flashes, one rightward, one leftward. Only C can have a clock-reading of this event at the moment of launch (since no other mover is there).
Event e_D: a flash reaches D's mirror. Only D can have a clock-reading of it.
Event e_B: a flash reaches B's mirror. Only B can have a clock-reading of it.
Event e_r: both flashes return to C. Only C can have a clock-reading of it.
Event $e_{1/2}$: event of the clock-reading by C half-way between e_l and e_r. This event cannot be clocked as it happens by C nor by the other movers. It can only be calculated later, after C has read its clock at e_r, for uniform motion means: *the event half way is the event half time*.

The flashes launched at e_l make, in the graph, 45° and -45° lines. That means they are parallel to those launched at origin O, since the flashes they depict have the very same speed of light. And this remains after the two flashes are mirrored and return. Thus, the flash paths form the red square in the graph.

Retracing event $e_{1/2}$ – "half way is half time" - is important to C because in C's perspective this home event $e_{1/2}$ is *simultaneous* to the away-events e_B and e_D. That means C's clock-reading of $e_{1/2}$ is, in C's perspective, the "time at which e_B and e_D happened".

At event e_r, C will know, by calculation, where B and D *were* halfway the time between C's clock-reading of e_l and of e_r. "Halfway time" means here: the half-way-clock-reading of C. C is chosen in Fig. 3 as reference. C is set as the "zero-mover". Simply since we measure distances from C. So in Fig. 3 events are ordered according to C's coordinates (x_C,t_C) where C measures its own position as the vertical (time) axis x_C=0 for all t_C. Arbitrarily so, for B and D could have been set x=0 with no less justification, which would give a different perspective.

By transforming the graph from the perspective of C to the perspective of B (from the left to the right graph in Fig. 4) you actually do the Lorenz-transformation. In such a transformation you shift to using B's coordinates (x_B,t_B) instead of (x_C,t_C) (the mathematics is for section 3).

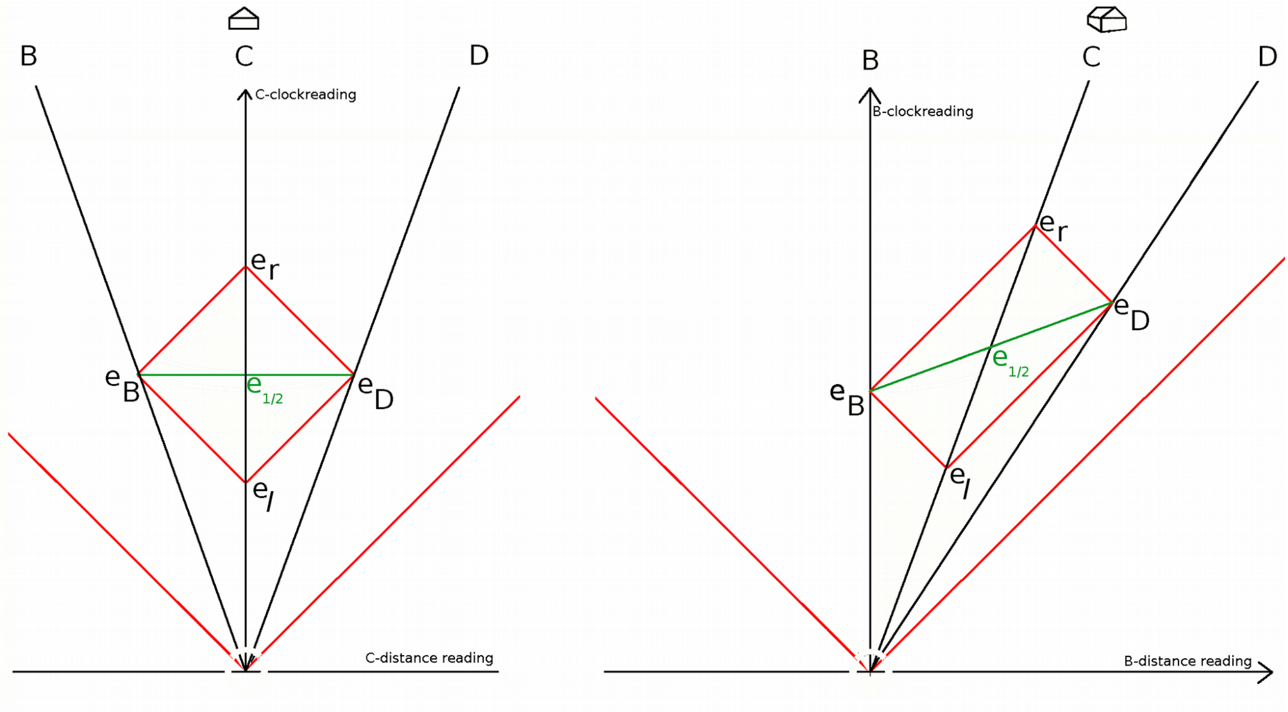

Fig. 4

In B's perspective the world line of B is vertical, since it is now B, instead of C, who zeroed his position. C and D move rightward relative to B so their world lines go up to the right. The result is that going from the *source* coordinate system (left) to the *target* coordinate system (right), all three world lines have rotated clockwise compared to the source, in such a way that the relative speed of C in the target system (that is: the speed v_{BC} of C in B's measurement) equals the value of v_{CB} in the source system, but negative:

$$v_{BC} = -v_{CB}$$

Since v_{CB} is a leftward speed (negative), the speed v_{BC} of C in the perspective of B (right graph) will be positive.

The light lines of the C's flash paths shifted to the right but did not rotate! For light speed is the same under all perspectives, including B's. Hence though B plots the others as moving at different angles (speeds), he plots the tracks of the light flashes at the same angles of 45°, as always in normalized Minkowski diagrams.

What governed the rotation of the world lines to make the right graph of fig. 4? The green numbers in the graphs in Fig. 5 show the steps of construction:

1. B's world line should now be set vertical. Since C's world line was the former vertical axis, the result for C is a horizontal flip ($v_{BC} = -v_{CB}$) : C's world line now slopes forward with the same slope as B's world line sloped up leftward in the previous graph.
2. starting from the flash-launch-event e_l, event e_B should be where the 45° light line from e_l cuts through the vertical axis (the world line of B).

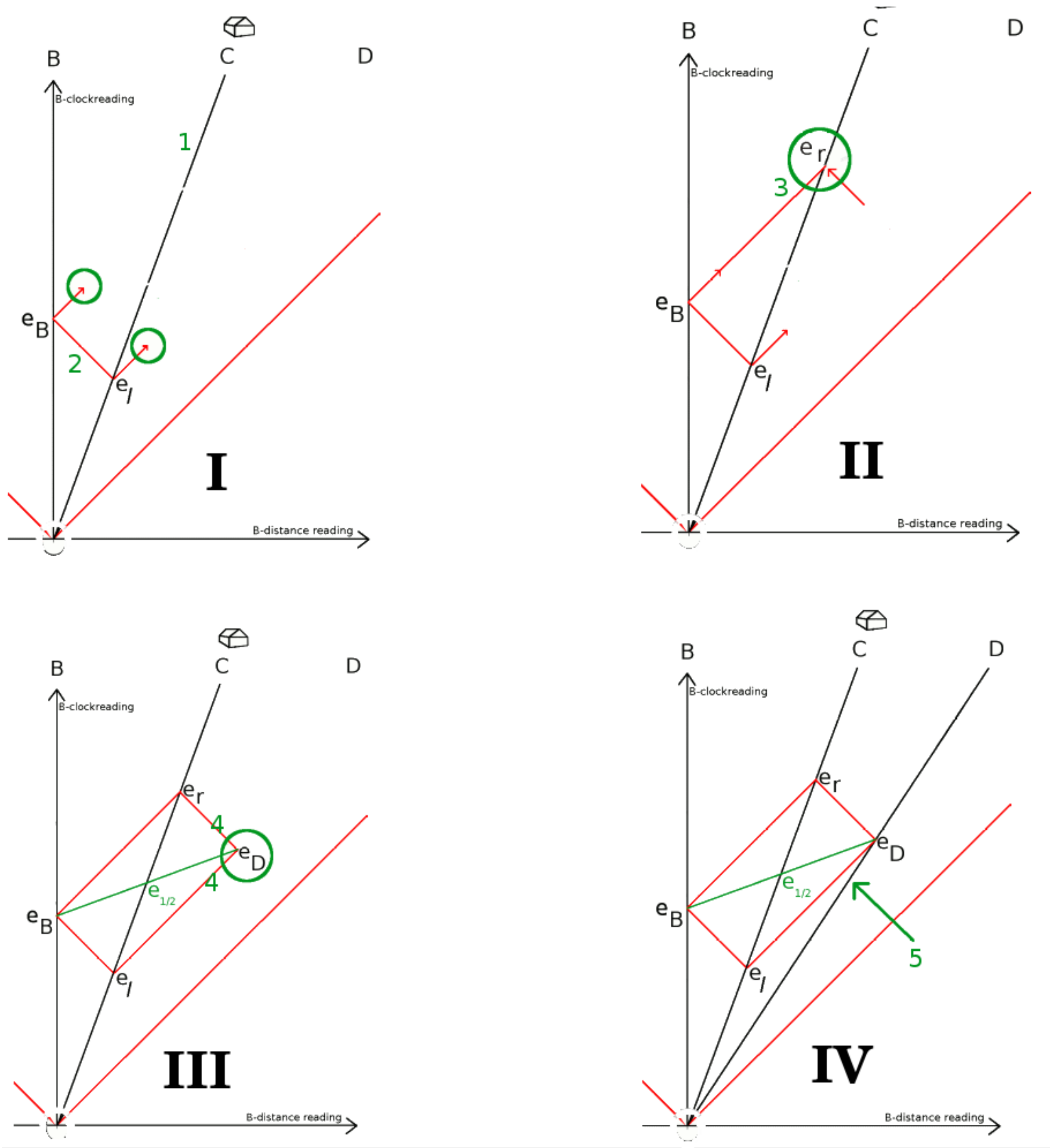

Fig. 5

3 e_r must be where the light line from e_B, with direction 45°, gets back at the world line of C, now sloping.

4. We know that exactly at event e_r, the other flash, the flash from D, returned to C as well. This fixes e_D by completing e_l – e_B - e_r as a rectangle. It should be a rectangle, due to, again, the 45° slopes of the light lines. Thus after transformation, event e_D should be exactly at the right most corner of the rectangle constructed.

5. e_D is a point on the world line of D, which should be a straight line (constant speed), hence D's world line should be the straight line through orgin O and e_D[3])

Going to this new B-perspective on the right side of Fig. 4, C's flash paths, that form a *square* in C's perspective, get squeezed into a *rectangle*. This means that in B's measurement, the light flash from C that B mirrors travels *longer* on the way back to C then it did when it went to B, that is, in its first (leftward) leg, which should be no surprise: from B's perspective, while the light makes its way from C to B and back to C, C moves away from B to the right, so the distance for the light to cover the way back to C is longer. From C's perspective this was not relevant since C calculates where B's mirror was at the moment of reflection. Where that mirror was before and after the reflection does not matter if you just want to plot the distance at the moment of mirroring, considering yourself at rest and the mirror as moving. In Fig. 4 right side, event e_B, B's own reflection of C's flash has an *earlier* B-clock-reading then e_D. While (Fig. 4 left side) in C's perspective e_B and e_D occur simultaneous. This obviously means C and B have different standards to establish which events are simultaneous: events simultaneous to C (or *C-simultaneous*) will not be simultaneous to B. Though obvious from this example, it will prove the hardest aspect of Lorentz transformation to absorb in your cerebral routines, but once you succeed, if you are like me, the whole issue will assume transparency (section 5).

In our graphical construction, while going from the C-perspective (left graph of Fig. 4) to the B-perspective (right graph of Fig. 4) the angle between the world lines C and D got visibly smaller, for same reason: the square got squeezed to a rectangle. This shows that in general calculating relative speed v_{BD} can not be done in the way of the hunter's brain, which is simply adding v_{BC} and v_{CD} (section 7).

There is symmetry over the vertical (time) axis in these transformations as shown when you analogously transform to D (setting D's time axis vertical) in Fig. 6 below, left graph. It is called *γ-symmetry* in the math of this matter, explained in section 3*)*

3) We graphically construed all except the transformed flash launch spot: event point e_l. Its place is arbitrary for the conclusions of this section: if you move e_l up and down along the C's world line (i.e. changing the launch-time), this only changes the size, not the aspect ratio of the resulting rectangle, hence would find (at its up-right corner) another point of that same line we construed (green number 5 in Fig. 5) as the world-line of D. Where exactly e_l lands by transforming to the B-perspective comes out easily in section 4 using the math of the hyperbola that derives from the assumptions graphically made in this section.

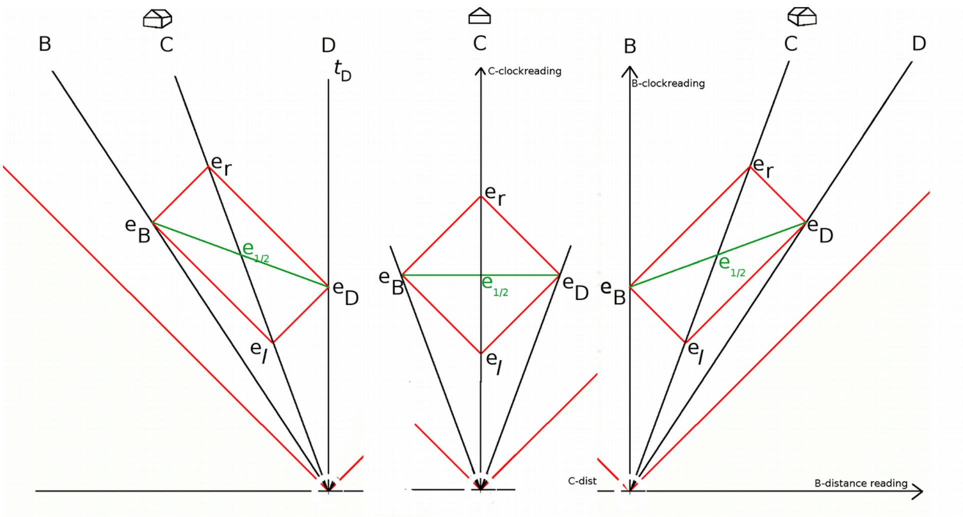

Fig. 6 “γ-symmetry” (math explained below section 3) illustrated

3. Deriving the equations of the Lorentz transformation.

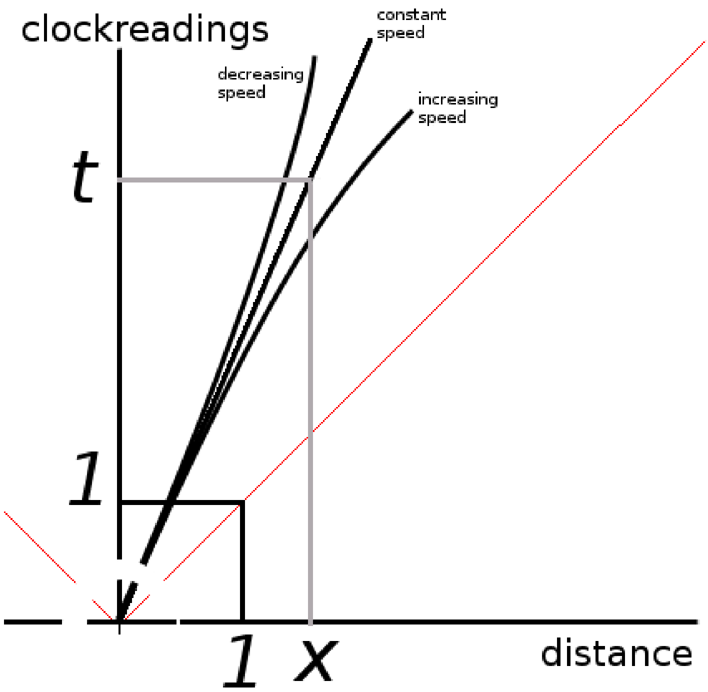

Fig. 7

Under a transformation as from the left graph to the right graph in Fig. 4, all event-points (x_C,t_C) , where x_C denotes mover C’s distance-reading of the event and t_C is C’s clock-reading, shift to another place hence acquire new, transformed, coordinates (x_B,t_B). The transformation equations

should give us the values (x_B,t_B), that is, the new position of every event-point in the transformed graph, as a function of the event point's old position (x_C,t_C).
In the perspective of one mover, the world line of another mover will be straight if that mover's relative speed v is constant ("uniform motion"). If another mover's relative speed is increasing or decreasing, its world line is a non-straight curve (Fig. 7). If we start finding the restrictions that the transformation we seek should satisfy, one of them must obviously be that uniform motion is transformed to uniform motion. That means that straight lines (constant speed) should stay straight under transformation. Hence transforming from C to B there must at least be fixed parameters α, β, γ, and δ such that the functions we try to find satisfy

$$x_B = \alpha t_C + \beta x_C \quad , \quad t_B = \gamma t_C + \delta x_C$$

The postulate of the absolute light speed yields further restrictions: the light pulses from the origin should in C's perspective have lines $x_C = t_C$ and $x_C = - t_C$. How does this measure from the B-perspective? B measures the same absolute light speed c=1, hence $x_B = t_B$ and $x_B = - t_B$. If the transformation should satisfy these requirements, these four equalities for light allow for the elimination two parameters as follows.

First consider the rightward flash from the origin: $x_C = t_C$ *and* $x_B = t_B$ should hold

In $x_B = \alpha t_C + \beta x_C$ substitute t_B for x_B and t_C for x_C to get $t_B = \alpha t_C + \beta t_C$ hence $\frac{t_B}{t_C} = \alpha + \beta$

In $t_B = \gamma t_C + \delta x_C$ substitute t_C for x_C to get $t_B = \gamma t_C + \delta t_C$ hence $\frac{t_B}{t_C} = \gamma + \delta$

hence $\gamma + \delta = \alpha + \beta$

Second, consider the leftward flash from the origin: $x_C = -t_C$ *and* $x_B = -t_B$ should hold

In $x_B = \alpha t_C + \beta x_C$ substitute $-t_B$ for x_B and $-t_C$ for x_C to get $-t_B = \alpha t_C - \beta t_C$ hence $\frac{t_B}{t_C} = -\alpha + \beta$

In $t_B = \gamma t_C + \delta x_C$ substitute $-t_C$ for x_C to get $t_B = \gamma t_C - \delta t_C$ hence $\frac{t_B}{t_C} = \gamma - \delta$

hence $\gamma - \delta = -\alpha + \beta$

Adding the blue equations yields: $2\gamma = 2\beta$ hence $\beta = \gamma$
Subtracting them yields: $2\delta = 2\alpha$ hence $\alpha = \delta$

Eliminating α and β yields

$$t_B = \gamma t_C + \delta x_C \quad , \quad x_B = \delta t_C + \gamma x_C$$

Another restriction allows for elimination of δ: C plots B's time and place (x_C,t_C) such that x_C/t_C = v_{CB}. Substitute $v_{CB}\,t_C$ for x_C in the right hand (x_B-) expression to get:

$$x_B=\delta t_C+\gamma v_{CB}\,t_C$$

By definition, B plots its own position for all t_B *as* x_B=0, hence in self-plottings $x_B=\delta t_C+\gamma v_{CB}\,t_C=0$ hence δ=-γv_{CB}. That should be in the general rule, for such a general rule should fit all special cases including this one. Substituting -γv_{CB} for δ, the transformation equations to find should finally satisfy:

$$t_B=\gamma(t_C-v_{CB}\,x_C) \quad , \quad x_B=\gamma(x_C-v_{CB}t_C)$$

Now we seem not to have made any progress to specification for we got a new parameter v instead of δ so we still have two of them: γ and v. Mathematically, had we run out of restrictions, transformations would be indeterminable since, though v can be measured, γ is as yet arbitrary and every choice of a value for γ would yield another coordinate system. There would be no way to single one coordinate system out.

But we still have the requirement that transforming to a positive speed should yield the same time coordinates as transforming to a negative speed of the same value (as illustrated in the vertical symmetry of the left and right graph in Fig 6): for movers B and D, if v_{CB} = – v_{CD} , as in our example, the transformation using v_{CB} and the one using v_{CD} should yield the same t-values (and the negative of the x-values).

This requirement means that if, as in the example of Fig. 6 middle graph, v_{CB} = - v_{CD} even though v_{CD} and v_{CB} have not the same (but each other's negative) values, transforming to the left and the right graph should yield (while the x-value changes sign), the *same t*-values (*γ-symmetry*):

$$t_D=\gamma_{CD}(t_C-v_{CD}\,x_C) \quad \text{should be equal to} \quad t_B=\gamma_{CB}(t_C-v_{CB}\,x_C)$$

symmetry should allow for substituting v_{CD} for v_{CB}

$$t_D=\gamma_{CD}(t_C-v_{CD}\,x_C)=t_B=\gamma_{CB}(t_C-v_{CD}\,x_C)$$

Hence in our special case of Fig. 6 γ_{CD} and γ_{CB} cannot differ. A trivial implication is this: since for two movers X and Y always v_{XY}=-v_{YX}, transformation back and forth between the two should have the same value for γ:

always $\gamma_{XY} = \gamma_{YX}$.

For our specific example (Fig. 6), v_{BC}=-v_{CB}=v_{CD}=-v_{DC}, we now have harvested these equalities: $\gamma_{BC} = \gamma_{CB} = \gamma_{CD} = \gamma_{DC}$.

The requirement of *γ-symmetry* fixes the relation between γ and v, and thus allows the elimination of γ. As follows: consider the *t*-transformation from B to C

$$t_C = \gamma_{BC}(t_B - v_{BC}\, x_B)$$

The inverse transformation of *t* and *x* from C to B is

$$t_B = \gamma_{CB}(t_C - v_{CB} x_C) \quad , \quad x_B = \gamma_{CB}(x_C - v_{CB} t_C)$$

nest this last transformation (C to B) in the B-to-C-transformation above it to get a transformation of t_C over t_B *back to itself*:

$$t_C = \gamma_{BC}(\gamma_{CB}(t_C - v_{CB} x_C) - v_{BC}\gamma_{CB}(x_C - v_{CB} t_C)) = \gamma_{BC}\gamma_{CB}(1 + v_{CB} v_{BC})(t_C - \frac{v_{CB} + v_{BC}}{1 + v_{CB} v_{BC}} x_C)$$

we already found $\gamma_{BC} = \gamma_{CB}$ (*γ-symmetry*) and $v_{BC} = -v_{CB}$ so we can substitute γ_{CB} for γ_{BC} and $-v_{CB}$ for v_{BC}.

$$t_C = \gamma_{CB}^2 (1 - v_{CB}^2)(t_C - \frac{v_{CB} - v_{CB}}{1 - v_{CB}^2} x_C) = \gamma_{CB}^2 (1 - v_{CB}^2) t_C$$

The division-expression is zero. Divide left and right by t_C to get

$$1 = \gamma_{CB}^2 (1 - v_{CB}^2) \quad \textit{hence} \quad \gamma_{CB} = \frac{1}{\sqrt{1 - v_{CB}^2}}$$

This γ-function of relative speed v moves up to infinity with positive and negative speeds v_{CB} approaching light speed.

There are many ways to derive the Lorenz equations. This version, basically existing in the literature is, in my perception, based most directly on reading the Minkovsky graphs.

Thus transformation from B to C depends on only one parameter: v_{BC}. Substituting the expression in v found for γ in the transformation equations yields the Lorentz transformation :

$$t_C=\frac{t_B-v_{BC}x_B}{\sqrt{1-v_{BC}^2}} \quad , \quad x_C=\frac{x_B-v_{BC}t_x}{\sqrt{1-v_{BC}^2}}$$ [4])

For this derivation we needed no assumptions other than the ones introduced in sections 1. and 2. [5])

4 **Classical notation:** in the classical derivations in the literature, notation is as follows: source variables are unprimed, target variables are primed, so instead of suffixes like B, C, D and CB as in x_B, t_B, v_{BC}, one uses t, $t´$,$t´´$, v, $v´$, $v´´$, $v´´´$ etc. Both the primed the suffixed notation have their strong and weak points. In the classical notation, often speed of light is retained as c such as to allow units yielding $c\neq1$, which is now known to yield no additional insights in this matter. All in all, In the classical notation, the Lorentz equations appear thus:

$$t'=\frac{t-\frac{v}{c^2}x}{\sqrt{1-\frac{v^2}{c^2}}} \quad , \quad x'=\frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}$$

5 About the **method of proof** used: In narrowing down the freedom of our transformation equations from the α, β, γ, δ– form to the Lorenz-form we consistently took our restrictions from special cases. They were: straight lines should map as straight lines, light lines should keep making 45° slopes, the world line of the transformation target mover should become the vertical axis, cases of same speed reverse movement should satisfy *γ-symmetry*. Einstein (Relativitätstheorie, 1916) p. 76 uses a similar method to explain the Lorentz equations to a general public and writes: "… brauchen wir nur einen bestimmten Wert … ein momentaufname … ". This type of method does not prove the absence of further restrictions that might turn the exercise inconsistent and a solution impossible. Such proofs should exist, though I failed to find any. A related question is: why "derive" the Lorentz transformation at all? Why not praise the inventor and rest on his laurels? As long as theories work without requiring questionable special adaptations to fit the data, who cares? A reason yet to do these things it is that the requirements that generate the Lorentz transformation do help to understand what the transformation is doing to the physical quantities involved.

4. Time dilation: while their coordinates are being transformed, event-points move along fixed hyperbolas that are clock-reading frontiers [6])

In the example above B,C and D are launched such that B moves leftward from C with the same constant speed as D moves rightward from C, and C thus remains in the middle between B and D. There could be an A moving uniformly such that B remains in the middle of A and C, on the other side an E such that D remains in the middle of C and E. And this chain of even movers A,B,C,D,E,F, ... each of them remaining in the middle of its neighbours (v_{AB}=v_{BC}=v_{CD}=v_{DE}=$v_{EF=...}$) could be extended indefinitely. Each of those even movers can be set as the reference, the zero mover with its clock-readings measured vertically, in measuring our example of C's two flashes mirrored by B and D. That will result in graphs displaying different A,B,C,D,E, … perspectives, but all of the very same four flash-events caused by mover C. Every stage of the animation on *http://asb4.com/relativity/Lorentz-rotation.gif* depicts the same flash operation, but from the perspective of a different mover in this set …, A,B,C,D,E, ... of even movers.

In Fig. 8 below the three graphs of Fig. 6 are merged:

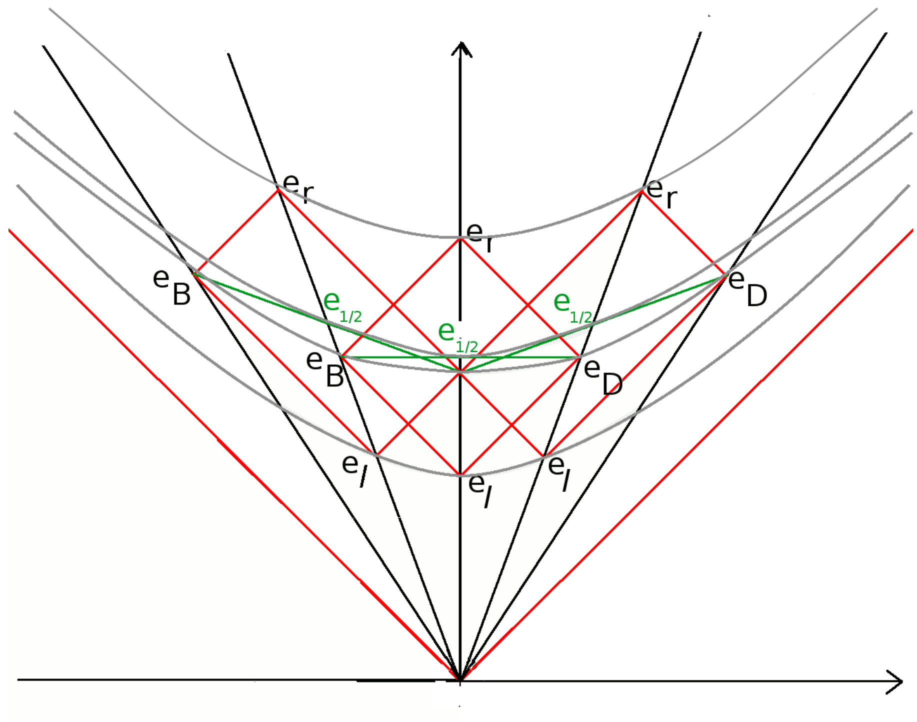

Fig. 8

The striking feature looking Fig.8 or, even more clear: at the animated graph *http://asb4.com/relativity/Lorentz-rotation.gif* is that the five events *e* plotted in the graphs, when being transformed to set any other mover to zero, each shift over a fixed curve (the grey lines), the

6 See appendix for the **basic math of the hyperbola**.

equations of which are, in fact, elementary hyperbolas. To derive them, transform, for instance, t_B to t_C (transforming from right to left in Fig. 4). The Lorentz equation reads

$$t_C = \frac{t_B - v_{BC} x_B}{\sqrt{1 - v_{BC}^2}}$$

In B's perspective, the equation of C's world line is $x_B = v_{BC} t_B$. To transform that world line to C's perspective substitute $v_{BC} t_B$ for x_B to get C's time scale in terms of B's time scale.

$$t_C = t_B \sqrt{1 - v_{BC}^2}$$

This transforms B's clock-reading t_B of all events *e* on C's sloped world line into C's clock-reading of that same event. The difference between these readings is known as *time dilation*. Since $-1<v<1$, in the B-perspective, C's clock runs slower than B's and hence reads *earlier* than B's ($t_C<t_B$).

Consider the clock-reading t_C, the one that is retraced by C for event $e_{1/2}$, and label it $t_C=\tau$: the equation transforming to value τ from the B- to the C-perspective is:

$$\tau = t_B \sqrt{1 - v_{BC}^2}$$

Now we return to the B-perspective (Fig. 9 below) and ask: what value t_B should B read on its clock for the away-event $e_{1/2}$ where C's clock reads τ? Solve for t_B:

$$t_B = \tau / \sqrt{1 - v_{BC}^2}$$

This plots, in the B-perspective, B's home clock-reading of the event *equitemporal* (happening at the same time, simultaneous) to C's home clock-reading event τ. $-1<v<1$. B's clock points *later* than τ.

Equation $t_B = \tau / \sqrt{1 - v_{BC}^2}$ finds the B's clock-reading for the remote event where C's clock reads τ. But since it depends solely on v_{BC} , generalized to movers X at all speeds $-1<v_{BX}<1$ as $t_B = \tau / \sqrt{1 - v_{BX}^2}$ you have the general relation between t_B and v_{BX} . By substituting x_B/t_B for v_{BX} , where x_B and t_B can now have any value, you see it is a hyperbola [7]):

$$\tau = t_B \sqrt{1 - \left(\frac{x_B}{t_B}\right)^2} \quad \text{hence (squaring left and right)} \quad t_B^2 - x_B^2 = \tau^2$$

This shows as in Fig. 9 (B-perspective):

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7) See appendix for the basic math of the hyperbola

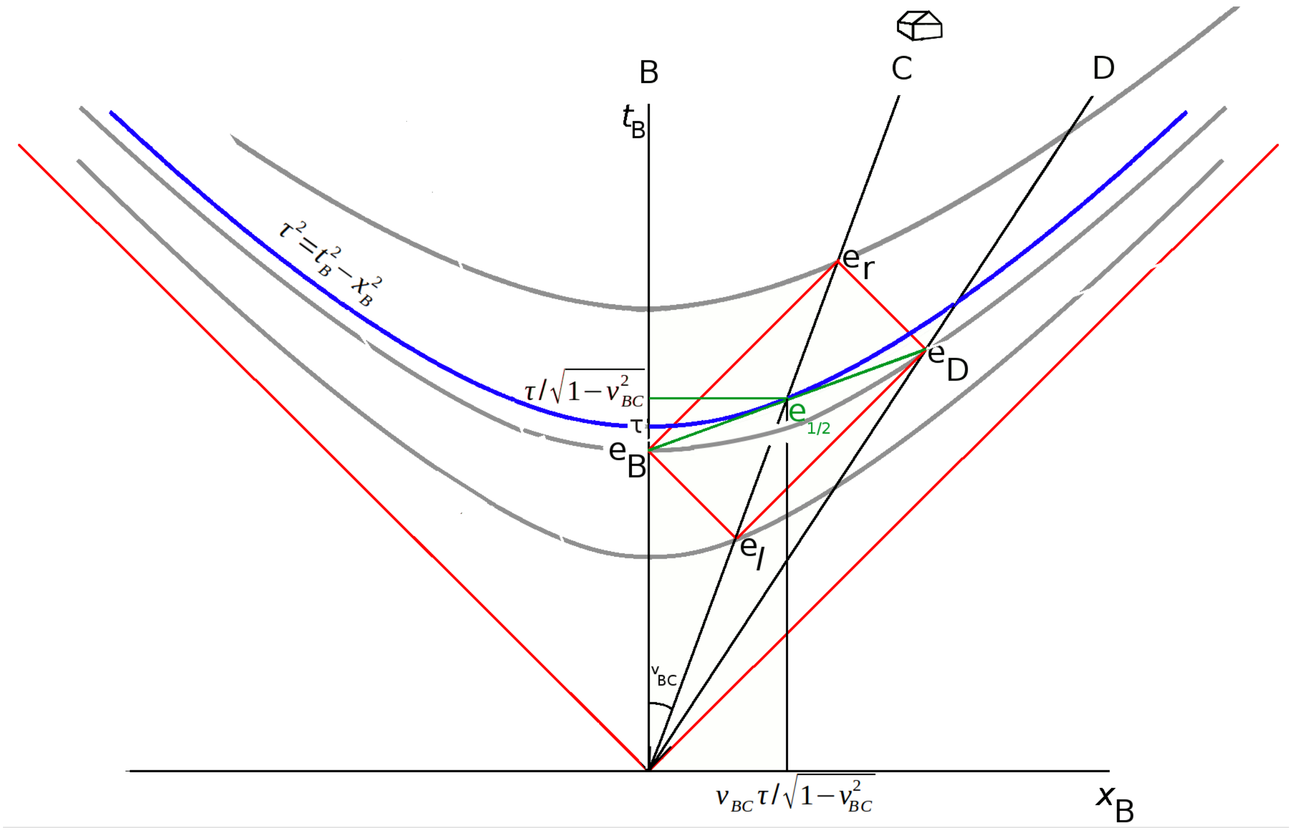

Fig. 9

On this hyperbola $t_B^2 - x_B^2 = \tau^2$ (blue in Fig. 9), where $x_B = 0$ you are at the vertical axis, on the world line of B and B's clock reads τ. In event $e_{1/2}$, C's clock reads τ. This event has, from the B-perspective, coordinates $x_B = v_{BC}\tau/\sqrt{1-v_{BC}^2}$ and $t_B = \tau/\sqrt{1-v_{BC}^2}$. That is a clock-reading *later* than τ. Generally, looking at this graph, C's world-line and thus its time grid is *stretched* compared to B's. Movers X with other speeds relative to B: $-1<v_{BX}<1$ have their world-lines stretched similarly and all read τ on their clock in the event (at the event-point) where their world line intersects with hyperbola $t_B^2 - x_B^2 = \tau^2$. For any mover X with speed v relative to B, B's clock-reading of the event where X's clock reads τ depends on v_{BX} only, and will be $t_B = \tau/\sqrt{1-v_{BX}^2}$.

We have arbitrarily set, as an example, τ as the time of $e_{1/2}$ *as being read by C* (on C's clock), which plots out the blue hyperbola. The three grey hyperbolas in Fig. 9 are, read from down upward, the one having e_l, the one shared by e_B and e_D, and the one of e_r. Generally: a higher (lower) value τ will produce a higher (lower) hyperbola $t_B^2 - x_B^2 = \tau^2$

Wherever you are on the blue hyperbola τ, you will always be at the event-point where the mover of the world line cutting through it exactly there, will read there τ on his clock. Inversely, it plots all positions that $e_{1/2}$ can get by applying all possible transformations $-1<v_{XY}<1$. Thus Fig. 9 the blue hyperbola is also the one over which $e_{1/2}$ moves if transformed (see animated graph http://asb4.com/relativity/Lorentz-rotation.gif). The hyperbola itself does not move under transformations. Transformations only move event-points along their fixed hyperbola, that is, in this example, the curve representing everybody's home-clock-reading τ.

So a hyperbola $t^2-x^2=\tau$ for some τ is the *clock-reading frontier* τ. *Under* the curve $t^2-x^2=\tau$ where $t^2-x^2<\tau$ all movers' clock-positions point *before* τ, *above* it ($t^2-x^2>\tau$), all mover's clock-positions point *after* τ. When you transform, all these clock-position τ event-points of all movers shift strictly over $t^2-x^2=\tau$.

This means that in Fig. 9 you see how, in the coordinates of the B-perspective, the time-grids of other movers, the distance between successive intersections of their world-line with the hyperbola's, stretch with speed. The time-grid is given by the distance between the marks of the "ticks" . This distance is larger with objects moving faster relative to B. Those marks follow the hyperbolas and, in B's coordinates (!) assume a wider distance for movers with speeds higher relative to B.

But you can transform to the coordinates of any such mover. Setting another world line vertical will result in rotating B's world line thus stretching B's time-grid in the same way. (see it happen again in the animated graph http://asb4.com/relativity/Lorentz-rotation.gif)

5. What happens "at the same time" is not the same for different movers

But *same clock-reading* and *same time* have become different things.

Keep in mind we are still considering leftward and rightward movements on one single line.

Our hunter's brain tempts us into wrongly thinking like this: at every point in time things happen on that line, so a little later something else may happen, a little earlier something else may have happened. But this wrongly ignores that different movers like B and C differ in their perception of which events happen at the same time (*simultaneity*) and which ones do not. Two events that happen at the same time *if measured from the C-perspective*, do not come out as happening at the same time *if measured from the B-perspective*. Each of them singles out a different set of events (a differently sloped straight line of points in the graph) as happening "at the same time". This, for a hunter, is a rather difficult thought-step to make.

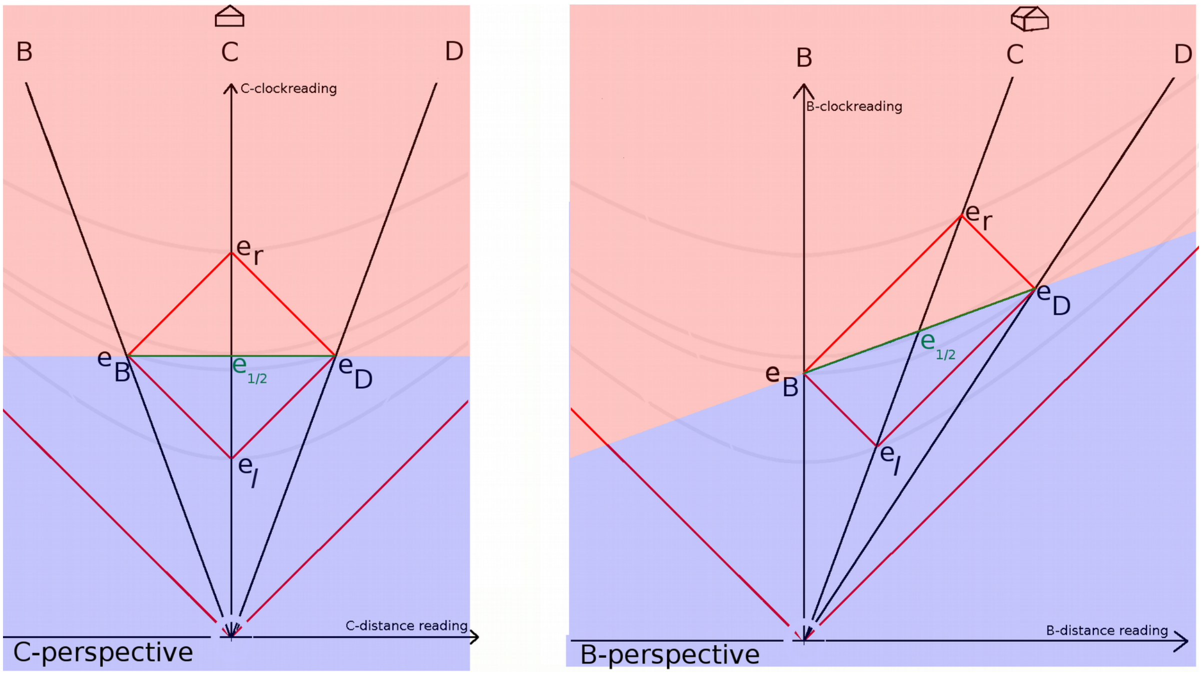

Fig. 10

In C’s perspective (Fig. 10 left graph), the flash-mirroring events e_B and e_D occur at the same time, and so do all other events on the horizontal borderline between the colours cutting through e_B and e_D , including $e_{1/2}$. In the C-perspective, events in the blue happen earlier, those in the pink happen later. The transformation to the right graph reshuffles all events. For these C-simultaneous events the result is that their line has rotated counterclockwise and now slopes up. The line still connects the points of events that occur at the same time *from C’s perspective*, but these events all got a new position. And what has not changed while transforming to the right graph is this: in the C-perspective, events in the blue – and those are the *same* events that are in the blue at the left side, they have just been reshuffled in the graph! - are *still* the ones that happen earlier *according to C*, those in the pink happen later *according to C*. In the right graph, however, it is B who measures simultaneity *horizontally*, and C’s simultaneity line is no longer horizontal. Hence, in the B-perspective the events on that upward sloping line of C-simultaneous events are *not* simultaneous. This is the result of the relative movement of B and C.

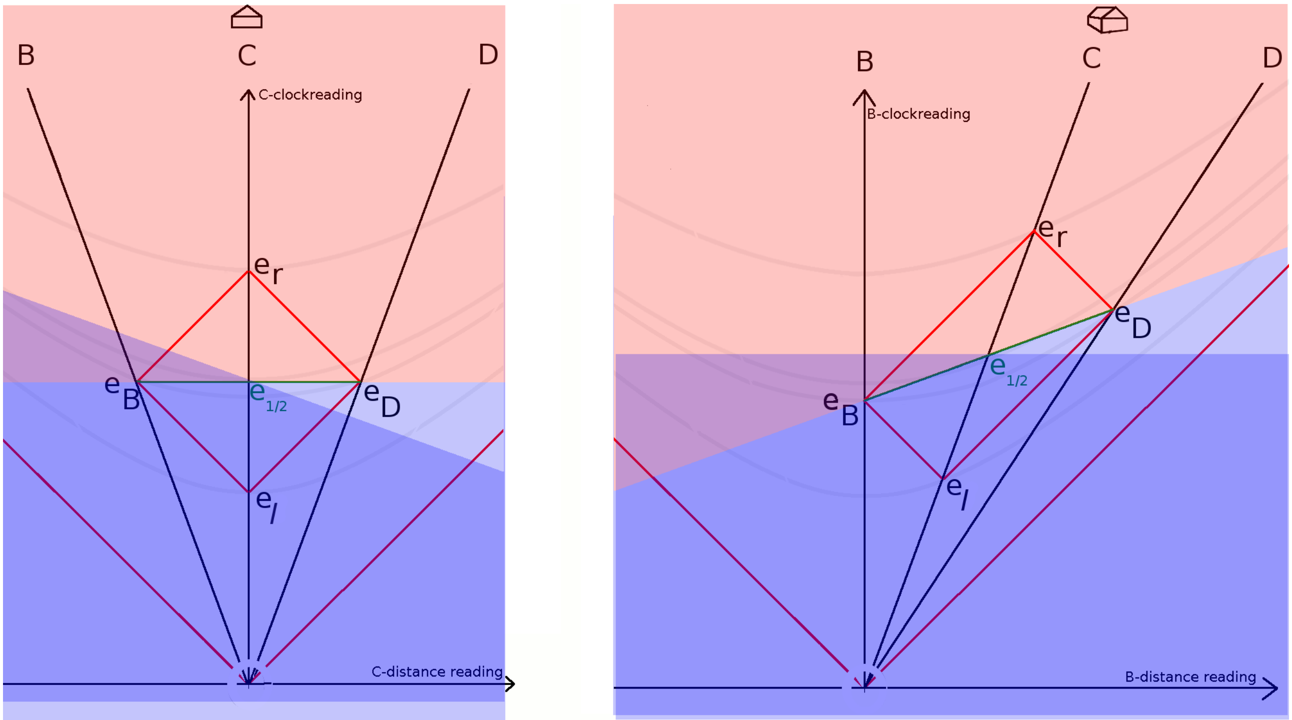

Fig. 11

This needs some effort to sink in. And it is crucial. The best exercise is to put equitemporal (or "simultaneity"- or "same time"-) lines of both B and C in in both graphs (Fig. 11). In the left graph the downward sloping borderline of colours is the B-simultaneity in the C-perspective. It becomes horizontal by transformation to the B-perspective (resulting in the right graph).

Transformation (left to right) is a reshuffling of all event-points. Events now are four-colour-tagged so we can see where they land after being reshuffled in the graph by transformation, choosing, as an arbitrary example, event $e_{½}$ as the reference (that is why all colours meet at $e_{½}$ both in the left and in the right graph):
pink: events later than event $e_{1/2}$ *both* from C's and from B's perspective
dark blue: events earlier than event $e_{1/2}$ *both* from C's and from B's perspective
purple: events *later* than event $e_{1/2}$ from C's *but earlier* than event $e_{1/2}$ from B's perspective
light blue: events *earlier* than event $e_{1/2}$ from C's *but later* than event $e_{1/2}$ from B's perspective

Here we took $e_{1/2}$ as the fixed event from which to analyze, but we could have done the same exercise using any other fixed event (of course with different outcome).

6. The math of the equitemporal lines, how they rotate under transformation

In a graph of the C-perspective you find all events C-simultaneous to $e_{1/2}$ on the horizontal line through $e_{1/2}$. To find them in another perspective, like that of B (Fig.12), label again as τ C's clock-reading at $e_{1/2}$. If τ should result from a transformation of time coordinates from B to C then B's source coordinates x_B,t_B should satisfy

$$\tau=\frac{t_B-v_{BC}x_B}{\sqrt{1-v_B^2}}$$

with τ and v_{BC} fixed, this makes a sloped line of points (x_B,t_B) in the B-graph

$$v_{BC}x_B=t_B-\tau\sqrt{1-v_{BC}^2}$$

It is a straight line with slope $1/v_{BC}$ (relative to the vertical axis) and constant $\tau\sqrt{1-v_{BC}^2}$. This constant is such that, where x_B=0, the line cuts through the vertical axis at B-clock-reading $t_B=\tau\sqrt{1-v_{BC}^2}$ (that is at event e_B).

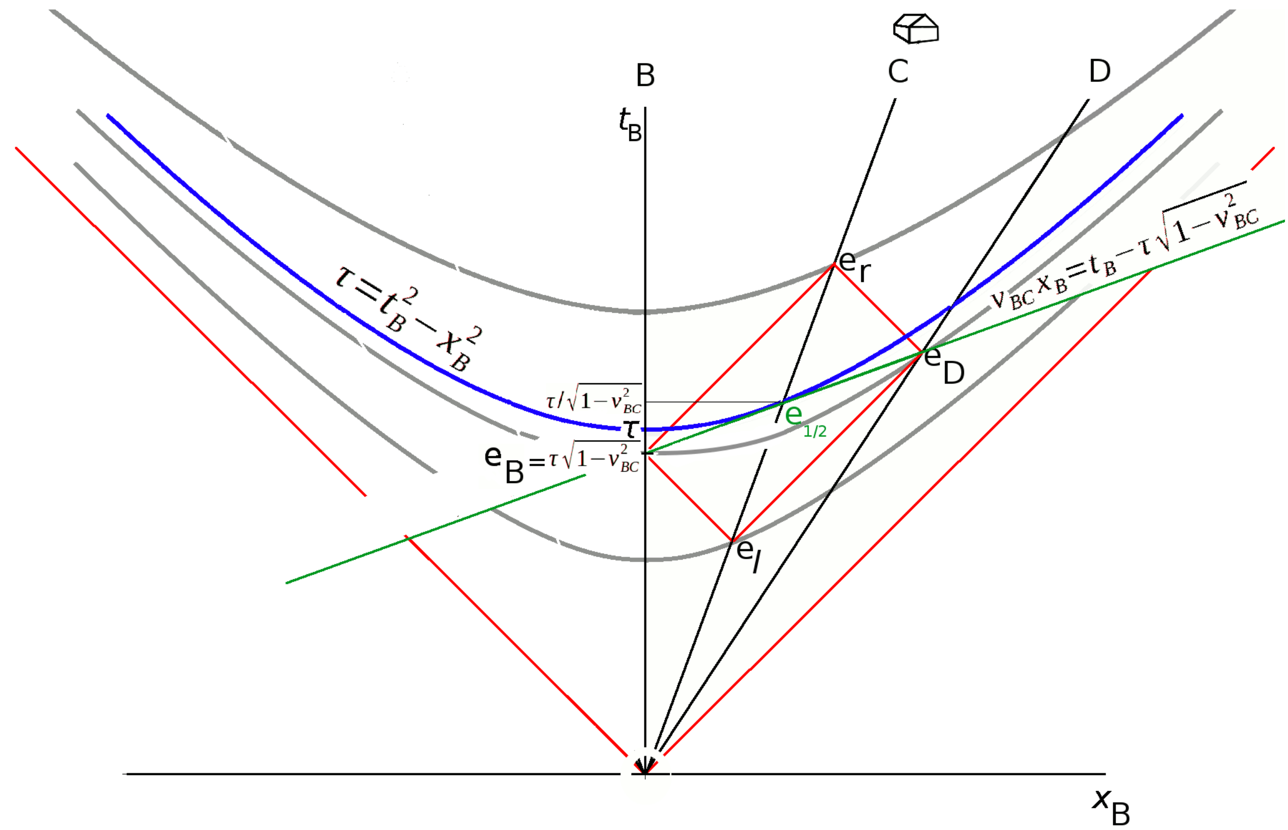

Fig. 12

This green sloped straight line in Fig. 12 is the same as the green lines in all previous graphs. It plots all B-coordinates x_B,t_B that give time τ if transformed to the C-perspective. It is, we can now prove, the tangent of hyperbola $t_B^2-x_B^2=\tau$ at event $e_{1/2}$: the local derivative of $t_B^2-x_B^2=\tau$ at its intersection with world line $x_B=v_{BC}t_B$ indeed has slope $1/v_{BC}$.

Proof: write $t^2 - x^2 = \tau$ as a function of *t*: $x = \pm\sqrt{t^2 - \tau^2}$. Take the positive side, apply chain rule of derivation:

$$\frac{dx}{dt} = \frac{d\sqrt{t^2-\tau^2}}{dt} = \frac{d(t^2-\tau^2)^{1/2}}{d(t^2-\tau^2)} \cdot \frac{d(t^2-\tau^2)}{dt} = \frac{1}{2}(t^2-\tau^2)^{-1/2} \cdot 2 = t/\sqrt{t^2-\tau^2}$$

With our B-perspective's subscripts this reads:

$$dx_B/dt_B = t_B/\sqrt{t_B^2 - \tau^2}$$

We need the value for dx_B/dt_B at $e_{1/2}$, where (see section 4), $t_B = \tau/\sqrt{1-v_{BC}^2}$. Substitute this expression for t_B in $t_B/\sqrt{t_B^2-\tau^2}$ to get, indeed, the slope $1/v_{BC}$:

$$\frac{dx}{dt} = \frac{\frac{\tau}{\sqrt{1-v_{BC}^2}}}{\sqrt{\frac{\tau^2}{1-v_{BC}^2} - \tau^2}} = \frac{1}{v_{BC}}$$

Both numerator and denominator have factors τ, canceling each other out. The remaining expression, in v only, simplifies to $1/v$. Hence for every mover v, the equitemporal lines of at the event-points of different clock-readings are always parallel so cannot intersect.

This helps to physically understand the rotation of the equitemporal lines: the line of C-equitemporal events, sloping if plotted in the B-perspective, turns out to be the tangent of the clock-reading frontier $t_B^2 - x_B^2 = \tau$ at the point where it intersects with C's world line ($x_B = v_{BC} t_B$). At that point the clock-reading frontier and the C-equitemporal line in B's perspective have the same direction/slope $1/v$ (measured from the vertical axis). Now focus on the zeroed mover: its equitemporal line is horizontal by definition. Thus a zeroed mover looking at neighbours with the clock-positions closest to his own will look, in its "own" graph, in horizontal direction. But a mover at another speed will find those equitemporal neighbours *along the local tangent of the clock-reading frontier*. For mover C this is the green line in Fig. 12, which hence represents the local simultaneity at the event $e_{1/2}$ we chose to focus on: the set of events locally measured as happening at the same time. That tangent rotates when you leave event $e_{1/2}$ to move along the concave clock-reading frontier.

The green line in Fig. 12 shows, in the B-perspective, the set of all event-points that in the C-perspective are measured as happening simultaneously with event $e_{1/2}$.

This means that e_B and e_D happen at the same time in the C-perspective, while in the B-perspective e_B happens earlier than e_D.

The straight C-equitemporal borderlines between the pink and blue areas in the right graph of Fig. 10 are described by the same equation $v_{BC}\, x_B = t_B - \tau\sqrt{1-v_{BC}^2}$

Since v_{BC} = - v_{CB} , thus $v_{BC}^2 = v_{CB}^2$ the C-equitemporal line be written with v_{CB} as well as with v_{BC}.

The case $v_{BC} = 0$, hence $t_B = \tau$ is the special case where C would be with B (not move relative to B) and the line would be horizontal.

Conclusion: consider in the B-perspective the event where C's clock reads τ. The set of events that C measures as *happening at the same time* as τ, is a straight line with, in the B-perspective, slope $1/v_{BC}$ (measured from the vertical axis). The slopes of all those straight lines (for all τ) are all parallel to one other.

Those parallel straight lines are mover C's equitemporal lines of all his home clock-reading events. They show how C times events.

If you consider ever larger relative speeds v_{BC}, the speed approaches light speed ($v \rightarrow$ c=1). Hence the slope $1/v$ rotates up (counterclockwise) to a limit $1/v \rightarrow 1$.

See the rotation process happen again in the animated graph *http://asb4.com/relativity/Lorentz-rotation.gif)*. This time focus on the rotation of the green diagonals in the red rectangles: they are sections of equitemporal lines. [8])

What, for some mover, is the exact slope of the line cutting through all points of events happening at the same time depends *only* on the speed of that mover relative to the zeroed mover: if you set yourself as the "zeroed-mover" and plot your own time on the vertical axis, the line of your simultaneous events is horizontal. If you zero a mover different from yourself with different speed, your equitemporal line will, if you now have become a relative rightward mover, be rotated counter clockwise (and clockwise if relative to the zeroed mover you move leftward). The events that your rotated line identify as simultaneous will be exactly the same events as they are under all other rotations, since these rotated exactly with that line. Relative to the mover now zeroed, these events are not simultaneous, nor will they under any transformation as long as there is a non-zero relative speed.

7. You can't perfectly compound speeds by adding them. The error gets nasty at speeds in the order of light speed

Let there be three movers B,C,D. We seek to know the error made by adding speeds as in $v_{BD} = v_{BC} + v_{CD}$, as our human hunting genes cause our brains to do (example section 1). We seek to know how the speed $v_{BD}=x_{BD}/\,t_{BD}$ relates to $v_{BC} = x_{BC}/\,t_{BC}$ and $v_{CD} = x_{CD}/\,t_{CD}$. The expression x_{CD} denotes distance of D in the perspective of C, we now have to double-index variables x and t in expressions like "x_{CD}", since we will now encounter more than one nonzero mover in more than one perspective.

8) For more complicated issues usually matrix notation is applied to transformations, which also makes for a smooth transition from movements along one line, as we analyze here, to movements in three space-dimensions.

To correctly find v_{BD} by compounding the speeds v_{BC} and v_{CD} , start with the Lorentz transformation from C to B, but only for C’s world line of D, that is only for source coordinates x_{CD} , t_{CD} :

$$x_{BD}=\frac{x_{CD}-v_{CB}t_{CD}}{\sqrt{1-v_{CB}^2}} \quad , \quad t_{BD}=\frac{t_{CD}-v_{CB}x_{CD}}{\sqrt{1-v_{CB}^2}}$$

Substitute these expressions for x_{BD} and t_{BD} in the formula $v_{BD}=x_{BD} / t_{BD}$ of the speed we seek to know:

$$v_{BD}=\frac{x_{BD}}{t_{BD}}=\frac{x_{CD}-v_{CB}t_{CD}}{t_{CD}-v_{CB}x_{CD}}$$

substitute – v_{BC} for v_{CB} to get

$$v_{BD}=\frac{x_{CD}+v_{BC}t_{C}}{t_{CD}+v_{BC}x_{C}}$$

divide numerator and denominator by t_{CD} to get

$$v_{BD}=\frac{\frac{x_{CD}}{t_{CD}}+v_{BC}}{1+\frac{x_{CD}}{t_{CD}}v_{BC}}$$

Substitute v_{CD} for x_{CD} / t_{CD}

$$v_{BD}=\frac{v_{BC}+v_{CD}}{1+v_{BC}v_{CD}}$$

This is flawless compounding of speeds. If you approximate this by simply adding the speeds as in $v_{BD} = v_{BC} + v_{CD}$, you round $1+v_{BC}v_{CD}$ to 1 (one).

The percentage overestimate of v_{BD} of assuming $1+v_{BC}v_{CD}=0$ is thoroughly irrelevant for the chances of the hunter to a hit in the buck-hunting case where B is a sitting spectator, C is a running hunter and D is a buck speeding away from the hunter: A good hunter’s sprinting speed relative to the spectator is something like 35 km/h. A good buck’s sprinting speed relative to the spectator should be 60km/h , *in the units used* (light speed= 299 792 458 km/sec is set to 1 (one)) the correct speed resulting is
0,0000000555555555555555
and the approximation by the hunter’s addition is
0,0000000555555555555556
the difference can just be caught by the last digit of an ordinary computer’s spreadsheet capacity. That is, in the unit 299 792 458 km/sec. You can forget seeing on a home PC screen an error ratio other than loads of nothing but zeros, or a non zero difference if you calculate in the units of km/sec let alone in m/sec.
It is not even among the worst of error risks if you’re determined – I never tried - to hit Pluto (speed 17000 km/h relative to the sun at 5.5 light hours), but if you absolutely want to hit it the first shot and you control more serious sources of error I could imagine you

may consider to respect it. I did not even read of a case where the calculation is used for launching missiles in our solar system. But Fig. 4 assumes speeds that illustrate the issue: in the right graph the angle between the world lines of C and D is visibly smaller than that between B and C, though they were equal in the left graph. This is because in transforming from C to another mover, C's light square gets squeezed to a rectangle due to the 45° requirement to the red lines. And the resulting squeeze determines the new positions of the three world lines. If you put a protractor on the right graph Fig 4, the angle of C to B's vertical axis measures about 20º, so v_{BC}= tg20º=.364. The angle between B and D' world lines is 33º, v_{BD}= tg33º=.649. In this example we graphically constructed the case where C is in the middle of B and D: in the left graph of Fig. 4 : v_{BC}= -v_{CB}= tg20º=.364. Adding those would yield v_{BD}= .728. That makes a 12% overestimate. What that error could cost you in money depends on your plans.

Another conclusion from the proper compounding of speed is this: since $-1<v_{BC}<1$ and $-1<v_{CD}<1$, compound speeds can not exceed the interval $-1<v_{BD}<1$. So the limits of infinitely compounding speeds of ever faster movers are $v = 1$ and $v = -1$. No speed higher than c=1 or lower than c = -1 can result from properly compounding speeds. Under proper compounding of speed, light speed is the limit (see it being approached, both by leftward and by rightward speeds, in the animation *http://asb4.com/relativity/Lorentz-rotation.gif*). The famous "maximum speed in the universe" follows directly from the Lorentz transformation.

8. Length contraction: movers measure shorter

The fact, confusing at first, that things when moving relatively to the observer measure shorter than when not moving, has the same explanation as time dilation (section 4). A 1m bar lying before you easily measures one meter because you clearly see where the left and the right end are at some point in time. As soon as the bar moves relative to you you'll have a hard time to determine the whereabouts of its two ends at *the same point in time*, a relativistic experiment you can do on your desk. Though good to ponder, reality is worse (section 5) than this joke: different movers single out different sets of events as simultaneous. What do "size" and "length" really mean, given the ways they are measured? You will need clock-readings to determine simultaneity of plotting both ends of an object. The Lorentz transformation explains it.

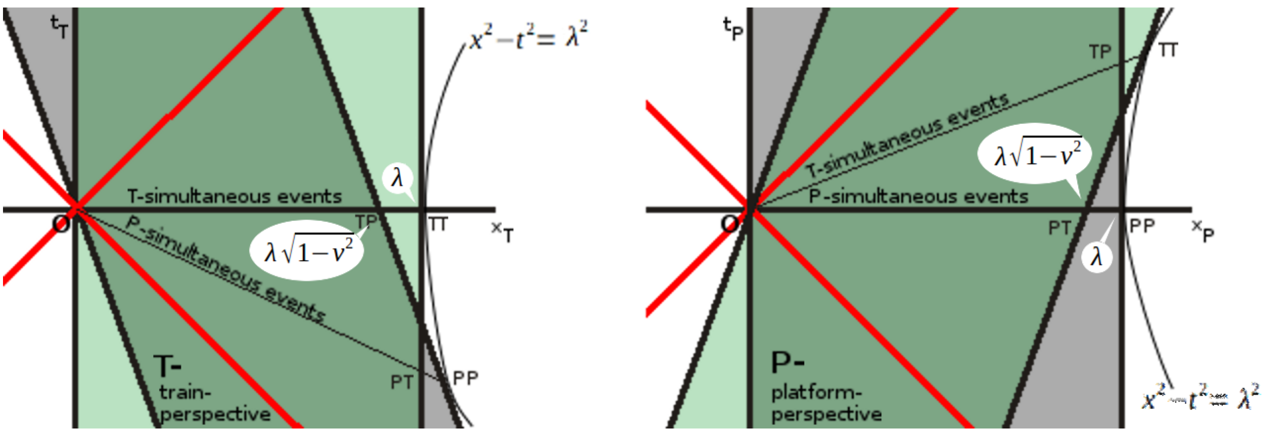

Fig. 13

Let relative speed be like in our previous examples. We have (Fig. 13) graphs like in Fig. 4, only instead of B and C we have a platform P and a train T. Instead of world-lines of event-points these two have coloured event-line sections (rear-to-front, left-to-right) that through time each leave as a trail a coloured strip, grey for the platform, green for the train. Around t=0, while the train passes the platform, the strips overlap for a while and this is what Fig. 13 focuses on. The right graph shows platform P‘s perspective. There, the train’s green strip leans rightward for that is where the train goes. Left, in the train’s perspective, the platform appears from the right and vanishes to the left.

The origin is defined as the event where the *rear* end of the train is at the *left* end of the platform and the clocks of both – both positioned at those left ends – are set t_P=t_T=0.

Let the length of both platform and train be λ in case the train is standing at the platform. This is called *rest length*. T and P will at all times measure that value for their own length, but the Lorentz equations imply that what they measure for each other’s length depends on their relative speed. In the left graph, you see at a distance λ from the origin the event point TT, which is the event of the front end of the train at time t_T=0. (“**T**rains front in the **T**rain’s perspective”). Keep in mind TT is the *single* front end event at t_T=0. The length the train measures for itself will at all other times, $t_T \neq 0$, be λ as well. Those measurements are (left graph) of all event points on the entire vertical line through TT, while we now consider only one of them, the event-point TT at t_T=0.

In the train’s perspective, TT must be at x_T=λ (left graph). In the right graph, at exactly that same place, in the P-coordinates, x_P=λ, you find PP, the event of the right end of the *platform* at time t_P=0. Where in the right graph is event TT? It should be somewhere, for it is an event, so its coordinates can be transformed. It is, as shown, up right, at a positive time (calculations next page). This means P measures event TT (that *same* event TT that the train – left graph - singles out for measuring the distance of its own front end) as taking place well *after* the moment the train’s rear end passed the origin. Now switch to the train’s perspective. From there, the platform, while correctly plotting on t=0 the rear-end of the train as x=0, by singling out event PT to measure the train’s front-end-distance, measures *too early*, hence the train too short. And this short story could be told in the reverse for TP. The measurement differences between the two movers are symmetrical.

In the right graph, the line of events P-simultaneous to t_P=0 is horizontal and the line of T-simultaneous events slopes up. Similarly, in T-perspective of the left graph the line of event P-simultaneous to t_T=0 slopes down. Those slopes cause the difference between the length measurements of the two movers. In the right graph, the line of T-simultaneous events slopes up with an angle of $1/v_{PT}$ relative to the vertical axis, as we derived in section 6.

This is how to calculate the difference in measured length: the Lorentz equation for transforming distance from right to left in Fig. 13 reads

$$x_T = (x_P - v_{PT} t_P) / \sqrt{1 - v_{PT}^2}$$

In particular, event TT should acquire as its x_T-coordinate: λ, the train’s own perception of its length. At t_T=t_P=0, the target-coordinate of this transformation, x_T, should equal λ. Hence the source coordinate of this transformation on the Platform side, (x_P,t_P)=$(x_P,0)$, should satisfy :

$$\lambda = x_P / \sqrt{1 - v_{PT}^2}$$

which fixes x_P, the length of the train in the platform's grid (right graph). This means that in the right graph the length x_P of the train in the platform's perspective, O-PT, must be shorter than λ, equal $x_P = \lambda\sqrt{1-v_{PT}^2}$ as the white balloon pointing at PT in the right graph indicates.

"Length contraction" actually is a pretty misleading term for a measurement result difference due to relative speed. Different distance perspectives arise because relative speed causes a difference between the movers in the *timing* of events. While we faithfully stick to the age old principle that length of moving objects is measured back-to-front *at the same time*, we found that two movers differ in singling out the front-events simultaneous to some back-event. Both "contractions" (TT to TP and PP to PT) can be seen in both graphs, left and right. Only the other side's numerical values can not be read at the axes. To get them you need to Lorentz-transform. Nothing contracts. Measurement results differ. My side's meters are longer, my seconds last shorter than the other side's *in my measurement*. Same for him. As long we move relatively to each other. That is all. Wish they told me right away.

In sum: 4 event-points (TT, TP, PP and PT) have 2 coordinates (x,t) in 2 perspectives. That makes 16 values. Applying the Lorentz equations between them results in the table below (check them each in Fig. 13).

Event	(x,t) in T-perspective (left graph)	(x,t) in P-perspective (right graph)
TT	λ , 0	$\lambda/\sqrt{1-v^2}$, $v\lambda/\sqrt{1-v^2}$
TP	$\lambda\sqrt{1-v^2}$, 0	λ , $v\lambda$
PP	$\lambda/\sqrt{1-v^2}$, $-v\lambda/\sqrt{1-v^2}$	λ , 0
PT	λ , -$v\lambda$	$\lambda\sqrt{1-v^2}$, 0

In both graphs, events PP and TT are on the hyperbola $x^2-t^2=\lambda^2$ (substitute the table values). That math is done for time $t^2-x^2=\tau^2$ (section 4), you only have to mirror $t^2-x^2=\tau^2$ over the 45º line x=t (see appendix on the math of the hyperbola – we just use λ instead of τ just to remember we are now talking distance instead of time). Thus $x^2-t^2=\lambda^2$ is the distance analogon of the clock-reading frontier (section 4): the *frontier of equal distance-reading*. From the origin, all movers read distance λ on hyperbola $x^2-t^2=\lambda^2$.

Finally focus on one such mover, and consider its distance measurements not only from the origin, at t=0, but at every point in time: the platform (as seen on both graphs of Fig.13, but easiest on the right side) will measure λ for its own length at all times as the width of the vertical grey strip, and, at all times $\lambda\sqrt{1-v^2}$, less than λ, for the train's length, for the back and the front of the train move together in uniform motion, so in the right graph the green strip while going up to the right does not change width. All the same the other way: neither does in the left graph the grey strip change width while receding .

You got it.

Appendix: math of the basic hyperbolas involved

A hyperbola of the type we used is a curve, a set of points in a flat plane. It has many equivalent properties that can be used as its definition. Here we choose the property of being a “conflict line”: set of points that all have the same distance to a given point and circle. Our point is F_1, our circle has centre F_2 and radius *r*.

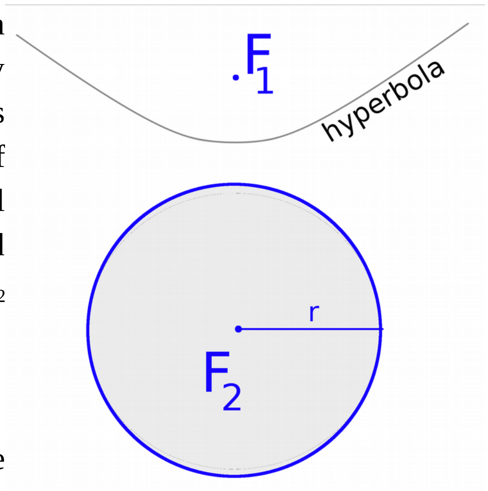

Fig. 14

This appendix shows this means that the hyperbolas featuring in Lorentz transformation are of type $t^2-x^2=\tau^2$

This is for time. When dealing with distance, time *t* and distance *x* swap position, which yields a flip (mirroring) over the *x*=*t* axis, as we shall see.

Fig. 15 is made out of Fig. 14 by drawing a circle with same radius *r* around F_1 as well, and installing an orthogonal grid using as the *x*-axis the straight line through the two intersection points of circles. The straight line F_1F_2 serves as *t*-axis. We can now baptize the *t*-coordinate of F_1 as *φ* and the minimum of the hyperbola curve as *τ*. And we can draw the downward mirror of the hyperbola curve: its maximum should be -*τ* .

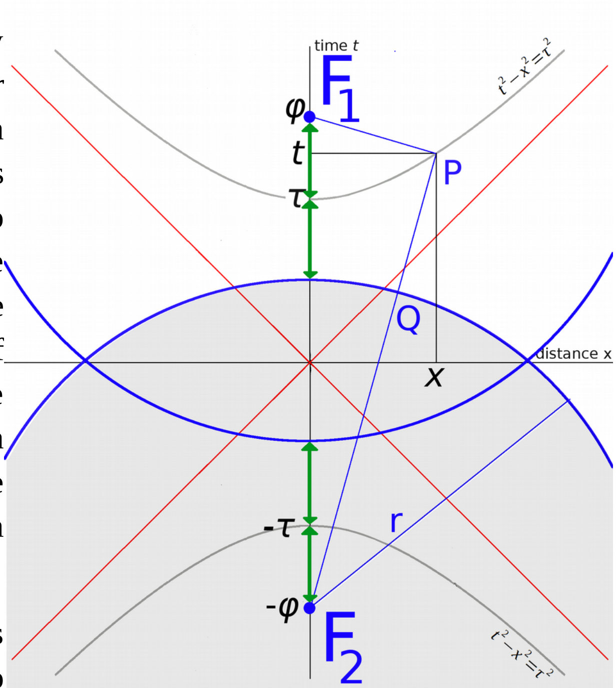

Fig. 15

Given the way the hyperbola is defined, the distance of point (*0*,*τ*) to F_1 should be equal to its distance to the circumference of the circle around F_2. *Mutatus mutandi* for (*0*,-*τ*). Hence the 4 distances marked with green double pointing arrows should all equal *φ*-*τ*.

Consider the *t*-axis section from the down-point (*0*,-*τ*) to the up-point (*0*,*τ*). Its length equals 2*τ*. Since the green subsections are all equally long, if you shift that entire section one green arrow up until the up-point is at F_1 and the down-point will exactly reach the circumference of circle F_1. Similarly if you shift the section down to F_2. Hence, for the radius of both circles: $r=2\tau$.

According to the definition of the hyperbola, the distance of any point P (x,t) on the hyperbola to F_1: $\sqrt{(\varphi-t)^2+x^2}$ should be equal to distance of P to Q. To get distance P to Q: the entire distance P to F_2 is $\sqrt{(\varphi+t)^2+x^2}$. Subtract the section F_2Q of this distance that forms the radius $r=2\tau$ of the circle. So the distance between P and Q is $\sqrt{(\varphi+t)^2+x^2}$ - 2τ

Hence the definition implies:

$$\sqrt{(\varphi-t)^2+x^2}=\sqrt{(\varphi+t)^2+x^2}-2\tau$$

Let us call this the *hyperbolic requirement.*
The rest is algebra. We have to prove that, if the hyperbola's asymptotes have 45° slopes, hence $x=\pm t$ as in our special Lorentz case of absolute speed of light, the hyperbolic requirement will simplify to $t^2-x^2=\tau^2$.

Proof:

The hyperbolic requirement is

$\sqrt{(\varphi-t)^2+x^2}=\sqrt{(\varphi+t)^2+x^2}-2\tau$ square to get

$$(\varphi-t)^2+x^2=(\varphi+t)^2+x^2-4\tau\sqrt{(\varphi+t)^2+x^2}+4\tau^2$$

$$\varphi^2-2\varphi t+t^2+x^2=\varphi^2+2\varphi t+t^2+x^2-4\tau\sqrt{(\varphi+t)^2+x^2}+4\tau^2$$

$$0=4\varphi t-4\tau\sqrt{(\varphi+t)^2+x^2}+4\tau^2$$

$\tau\sqrt{(\varphi+t)^2+x^2}=\varphi t+\tau^2$ square again to get

$$\tau^2((\varphi+t)^2+x^2)=(\varphi t+\tau^2)^2$$

$$\tau^2(\varphi^2+2\varphi t+t^2+x^2)=\varphi^2 t^2+2\varphi t\tau^2+\tau^4$$

$$\varphi^2\tau^2+2\varphi t\tau^2+t^2\tau^2+x^2\tau^2=\varphi^2 t^2+2\varphi t\tau^2+\tau^4$$

$$\varphi^2\tau^2+t^2\tau^2+x^2\tau^2=\varphi^2 t^2+\tau^4$$

$$\tau^2(\varphi^2-\tau^2)=t^2(\varphi^2-\tau^2)-x^2\tau^2$$

$$\tau^2=t^2-\frac{x^2\tau^2}{\varphi^2-\tau^2}$$

$$\frac{t^2}{\tau^2}-\frac{x^2}{\varphi^2-\tau^2}=1$$

Asymptotes: use again

$\tau^2(\varphi^2-\tau^2)=t^2(\varphi^2-\tau^2)-x^2\tau^2$ move all x- and t-factors left and expressions with only parameters to the right

$t^2(\varphi^2-\tau^2)-\tau^2 x^2=\tau^2(\varphi^2-\tau^2)$ divide by t^2:

$$\varphi^2 - \tau^2 - \tau^2 (x/t)^2 = \frac{\tau^2(\varphi^2 - \tau^2)}{t^2}$$

Now check *t* to infinity: we made the right side so as to have only parameters (constants) in the numerator so it will get zero. So the left side should go to zero as well. In that limit

$$\varphi^2 - \tau^2 - \tau^2 (x/t)^2 = 0$$

$$x/t = \pm(\varphi - \tau)/\tau$$

In case of 45° asymptotes through the origin as made by the Minkowski light lines $x=t$ hence $x/t=1$:

$$x/t = \pm(\varphi - \tau)/\tau = 1$$

$$\varphi = \pm 2\tau \quad \text{hence} \quad \varphi^2 = 2\tau^2$$

$\varphi^2 = 2\tau^2$ is a special case where $\frac{t^2}{\tau^2} - \frac{x^2}{\varphi^2 - \tau^2} = 1$ simplifies to

$$t^2 - x^2 = \tau^2$$

In dealing with lengths λ the variables *x* and *t* swap positions and the whole shit flips over the $x=t$ line (45° line)

$$x^2 - t^2 = \lambda^2$$

See Fig. 16. If needs be to get the point, practice flipping (mirroring) the graph of $y=x^2$ over the 45° $x=y$-line to draw $x=y^2$ while keeping the *x*- and *y*-axis on their places.

There you go.

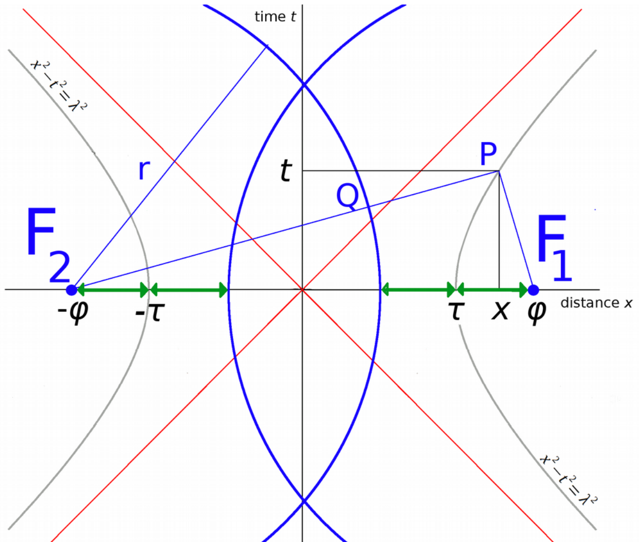

Fig. 16