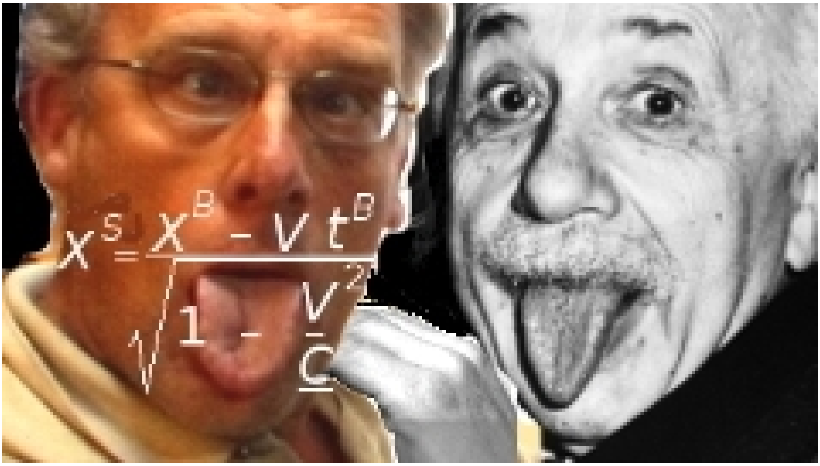
Bert **T**ells **W**hat **H**e **R**eads



Bert reads Einstein

Special Relativity Part 1 (/2)
Doing the Math

Bert[1] reads Einstein

Special Relativity Part 1 (/2)
Doing the Math

1. Where finiteness of light speed matters, applying Newton leads to serious errors

In cases where the finiteness of light speed does not matter, Newton can be applied to analyze objects passing at different speeds *v*. We can for instance treat two trains driving at constant speed, hence using energy only to offset friction, like bodies in uniform motion.

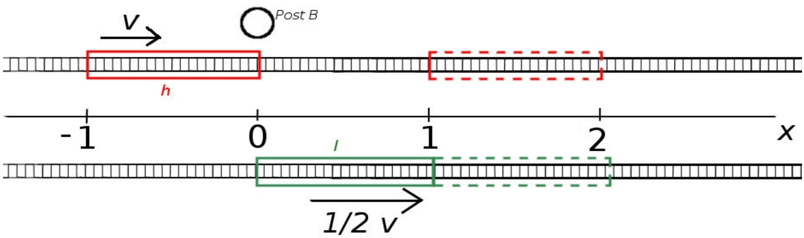


Fig. 1

The trains in *Fig. 1* have equal lengths that we set as the distance unit. As time unit υ we set the time the red high speed train *h* needs to make good its own length, so *h* has speed v=1 υmsec^{-1}. The green local train has half the speed which thus is v=½υmsec^{-1}. We set *t*=0 where the front of *h* and the back of *l* just arrive at Post *B*. That event is drawn by the solid red and green rectangles. Then at *t*=2 the situation will be as drawn by the dotted rectangles. Timing of the speed of *h* passing *l*,

1 Freeware pdf. This is the “self-summary” of my study. I used Einstein, A., *Uber die spezielle und die allgemeine Relativitätstheorie*, Braunschweig: Vieweg 1956, and Utrecht University's fall 2015 physics and astronomy bachelor's course in special relativity by Prof. Stefan Vandoren. About me: http://asb4.com/aboutme.html

whether you time from *h*, from *l* and from Post *B*, it all gives ½υsec , the speed of *h* minus the speed of *l*. No problems (they will come!).

Fig 2 depicts this in a space-time diagram.The lines represent the front points of the trains. Event e_0 is that of the front of *h* being at post *B*, event e_4 is that of the front ends being together at 2υm. Line *h* is 45° since v_h is set one.

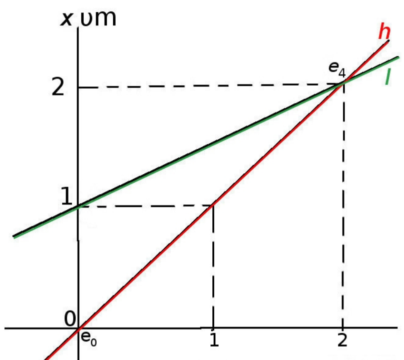


Fig.2

This introduces the important notion of an **event e**. Note that an event i ***"is"*** not a pair (t,x). It ***has*** a pair (t,x). For instance event 1, e_1, ***has*** $(t,x)=(0,1)$. Coordinates are ***properties*** of events.

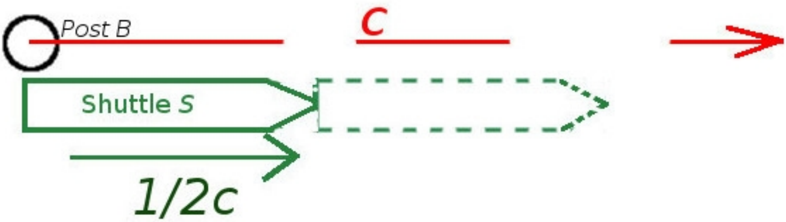


Fig. 3

In Fig. 3 we are in space. It is an area remote from mass concentrations and magnetic sources. Post *B* is now a space base, *S* (green) is a shuttle. Neither is using propulsion, so regard themselves for al practical purposes in uniform motion. *S* passes *B* with a relative speed equal to half the speed of light = ½*c* (*c*=299,792,458 msec^{-1}). *A* light pulse, shot by *B* at the event of the rear of the shuttle being at the post, is substituted

for the highspeed train. Both *S* and *B* set time and distance zero at that event.
From there and then, the light pulse goes on its way to overtake the shuttle and reach its front point. But where and when? Even now we have no rail there is an unambiguous answer but it requires Lorentz transformation and all the physical arguments that Einstein produced to start using it for general space-time measurement.

Here's why: you would expect *S* to measure the pulse passing with 1/2c, after all, its own speed relative to the "gun" *B* is 1/2*c*. But S measures 299,792,458 $msec^{-1}$. And that is not just in this example: *all* movers along the line of *any* light pulse, slow and fast, going the same way or in opposite direction, measure 299,792,458 $msec^{-1}$.

Well, light is often thought of as kind of a wave too, which reminds of "speed of sound" thinking. There, for the speed of the sound pulse, the speed of the pulse generator does not matter - but the speed of the detector does. In the case of measuring light speed not only the speed of the generator is irrelevant but that of detector is irrelevant too! With sound waves, the medium determines the speed of propagation. But unlike sound waves, light is not a medium-thing. There's nothing there. Light does a Doppler effect but that does not affect light speed. The only thing we have, relative speed of source and detector, does not yield any measured light *speed differences*. Both measure that same 299,792,458 $msec^{-1}$. Odd but true. Weird stuff, light.

2. How to accept light as it is?

In 1905 Einstein published the first article on how to accept light as it is. If light speed is the same for any two uniformly moving objects, then they must measure each other's meters and seconds as of different length. This would allow for the numerical value of the light's meters-per-second being the same among all movers, while the difference in meters and seconds between two moving systems could account for their relative speed. So “home-” meters and seconds must differ from “away-” meters and seconds.

In a single case such a thing can easily be forged. You could let meters differ, or the seconds, or a combination. If that's all allowed, you will have infinite ways to make light speed equal in two systems. But! There are requirements:

Requirements for transformation:
(i) proportionality under v : Under the transformation, at some constant (uniform) relative speed *v*, times and distances measured from *B* and *S* no longer need to be the same, as under Newton, but should have a *fixed ratio*, only depending on *v*, both if *B* and if *S* do measurements in the two possible directions on the line of their relative movement.

Proportionality under v can be generalised from one to three distance dimensions. But we turn out able first to find the transformation and do that generalisation only afterwards.

(ii) symmetry: The transformation should yield exactly the same result if you swap the labels *B* and *S* (or ¨home¨ and ¨away¨) for there is no way to choose between the swapped result and the original one.

These two requirements define a purely mathematical problem: there could be no such transformation, or there could be many. Or, as Einstein following Lorentz proved mathematically, there is only one: the *Lorentz transformation*.

3. Doing the math of the Lorentz transformation

Fifty years later, in 1956, Einstein[2†] could explain the math of his line of thought much easier to himself and others than in his first expositions, and probably rarely got back to his first cumbersome versions.

Post *B* (*Fig. 3*) measures the speed of the light pulse it generates when *S* passes as

$$c = \frac{x^B}{t^B} \qquad (=3\text{x}10^8 \text{msec}^{-1}) \qquad \Rightarrow$$

$$\Rightarrow \qquad x^B - ct^B = 0 \tag{1}$$

Our math assignment requires that, measured from S, light speed is the same c:

$$x^S - ct^S = 0 \tag{2}$$

For light travelling in the opposite direction (NOT, as you read in Einstein (1956) ”längs der negative X-Achse sich fortplantzenden Lichtstralen”[†]) speed is minus *c*

$$\frac{x}{t} = -c \tag{3}$$

$$(3) \Rightarrow \qquad x^B + ct^B = 0 \tag{4}$$

$$(3) \Rightarrow \qquad x^S + ct^S = 0 \tag{5}$$

For this math exercise, our assignment is to stay in the single dimension of the line on which *B* and *S* move away from (or towards) each other. So we analyse light speed in those same two directions (*Fig. 3*) and thus (1)(2)(4)(5) is all we consider as far as directions are concerned.

$$(1)(2) \Rightarrow \textit{ There is a } \lambda \textit{ such that } \left(x^B - ct^B\right) = \lambda\left(x^S - ct^S\right) \tag{6}$$

Similarly:

$$(4)(5) \Rightarrow \textit{ There is a } \mu \textit{ such that } \left(x^B + ct^B\right) = \mu\left(x^S + ct^S\right) \tag{7}$$

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2 Einstein, A., *Uber die spezielle und die allgemeine Relativitätstheorie*, Braunschweig: Vieweg 1956

(6) and (7) are weaker than their respective origins (1)(2) and (4)(5): they are necessary but not sufficient for their origins to hold simultaneously

We define *a* and *b*:

$$a := \frac{\lambda + \mu}{2} \qquad b := \frac{\lambda - \mu}{2} \qquad \text{(8a)(8b)}$$

(6)(7)(8) $\Rightarrow$ $x^S = ax^B - bct^B$ (see Lorentz workouts) (9)

(6)(7)(8) $\Rightarrow$ $ct^S = act^B - bx^B$ (see Lorentz workouts) (10)

Parameters *a* and *b* can be expressed in *λ* and *μ* (8), λ and μ are set usiing (6)(7) by (c, x^B, t^B, x^S , t^S).

So we can substitute all four, λ, μ, *a* and *b,* in the six-equation set (6)(7)(8a)(8b)(9)(10) and get a two-equation set without λ, μ, *a* and *b*,, that is, with only (c, x^B, t^B, x^S , t^S). For mathematical convenience we define a sixth variable, dependent on the other five (c, x^B, t^B, x^S , t^S):

$$v = \frac{b}{a}c \qquad \text{(see Lorentz workouts)} \quad (11)$$

(which by happy coincidence neatly shall denote the *relative speed of S and B* once after having done the math our pure math variables will start to refer again to physical quantities, see Lorentz workout (11)).

Define, for events e_i and e_j

$$\Delta x_e^S := x_i^S - x_j^S \qquad \Delta x_e^B := x_i^B - x_j^B$$

Proportionality under v (section 2) $\Rightarrow$

$$\textit{For all } i \quad \left[\frac{\Delta x_i^S}{\Delta x_i^B} \textit{ is the same} \right] \qquad (12)$$

(remember we consider only events e_i on the line on which *S* and *B* move away from each other)

Restriction (12) enables us to derive that ratio, fixed under *v*, from *any* position.

Measuring from B ...

Any position! That considerably simplifies the math: we can choose an easy time-point for *B* to derive a distance. t^B=0 obviously is such an easy one:

(9)(t^B=0) $\Rightarrow$ $$x^S = ax^B \tag{13}$$

(12)(13) $\Rightarrow$ $$\textit{For all } i \quad \left[\frac{\Delta x_i^B}{\Delta x_i^S} = \frac{1}{a}\right] \tag{14}$$

Under (14), when $a \neq 1$, *S*-meters are no longer *B*-meters.

Measuring from S ...

The easiest time-point for *S* to measure distance is t^S=0:

(10)(t^S=0) $\Rightarrow$ $$act^B = bx^B \quad \Rightarrow \quad t^B = \frac{b}{ac}x^B$$

(15)

(15)(9) $\Rightarrow$ $$x^S = ax^B + bc\frac{b}{ac}x^B \quad \Rightarrow$$

$$\Rightarrow \quad x^S = a\left(1 + \frac{1}{a}bc\frac{b}{ac}\right)x^B \quad \Rightarrow$$

$$\Rightarrow \quad x^S = a\left(1 + \frac{b^2}{a^2}\right)x^B \quad \Rightarrow$$

$$\Rightarrow \quad x^S = a\left(1 + \frac{b^2c^2}{a^2}\frac{1}{c^2}\right)x^B \quad \Rightarrow$$

with (11)$\Rightarrow$ $$x^S = a\left(1 + \frac{v^2}{c^2}\right)x^B \tag{16}$$

$$(12)(16) \Rightarrow \quad \textit{For all } i \quad \left[\frac{\Delta x_i^S}{\Delta x_i^B} = a \left(1 + \frac{v^2}{c^2} \right) \right] \qquad (17)$$

Symmetry (section 2) requires the transformation to be such that if you swap *S* and *B* consistently in the entire story, the result should be the same. Hence another condition for our transformation is:

$$(14)(17)(\textit{symmetry requirement}) \Rightarrow \quad \frac{1}{a} = a \left(1 + \frac{v^2}{c^2} \right) \quad \Rightarrow$$

$$\Rightarrow \qquad a = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} \qquad (18)$$

$$(11) \Rightarrow \qquad a = \frac{bc}{v} \qquad (19)$$

$$(18)(19) \Rightarrow \qquad \frac{bc}{v} = \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} \qquad \Rightarrow$$

$$\Rightarrow \qquad b = \frac{v}{c} \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} = \sqrt{\frac{1}{\frac{c^2}{v^2} \left(1 - \frac{v^2}{c^2} \right)}} \qquad \Rightarrow$$

$$\Rightarrow \qquad b = \sqrt{\frac{1}{\frac{c^2}{v^2} - 1}} \qquad (20)$$

$$(9)(18)(20) \Rightarrow \quad x^S = x^B \sqrt{\frac{1}{\frac{c^2}{v^2} - 1}} - ct \sqrt{\frac{1}{\frac{c^2}{v^2} - 1}} \qquad (21)$$

(10)(18)(20) $$x^S = x^B\sqrt{\frac{1}{1-\frac{v^2}{c^2}}} - ct\sqrt{\frac{1}{\frac{c^2}{v^2}-1}} \qquad (22)$$

(21) and (22) form the Lorentz transformation.

It usually is presented by defining γ:

$$\gamma := \sqrt{\frac{1}{1-\frac{v^2}{c^2}}} \qquad (23)$$

then the Lorentz transformation reads:

(21)(23) $\Rightarrow$ $$x^S = \gamma\left(x^B - vt^B\right)$$ (see Lorentz workouts) (24)

(22)(23) $\Rightarrow$ $$t^S = \gamma\left(t^B - \frac{v}{c^2}x^B\right)$$ (see Lorentz workouts) (25)

4. Back to physics

The Lorentz transformation thus is proven to be the *only* transformation satisfying the requirements of section 2, *proportionality under v* and *symmetry.* It implies that away-meters are smaller than home-meters en away-seconds last longer than home seconds. We made the transformation for *B*=home and *S*=away. Yet it is a neutral account. The only thing to choose is what to take as the “home” or unit side. This choice is arbitrary. Observers from *S* an *B* can exchange neutral information in terms of either after simply agreeing on a choice of “side”.

5. LORENTZ WORKOUTS

Workout (9)

$$\left(x^B - ct^B\right) = \lambda\left(x^S - ct^S\right) \qquad \text{repeats (6)}$$
$$\left(x^B + ct^B\right) = \mu\left(x^S + ct^S\right) \qquad \text{repeats (7)}$$

(6) + (7) $\Rightarrow$ $$2x^S = (\lambda+\mu)\,x^B - (\lambda-\mu)\,ct^B \qquad \Rightarrow$$

$\Rightarrow$ $$x^S = \frac{(\lambda+\mu)}{2}\,x^B - \frac{(\lambda-\mu)}{2}\,ct^B \qquad \Rightarrow$$

$\Rightarrow$ $$x^S = a\,x^B - bct^B \qquad \text{repeats (9)}$$

Workout (10)

(6) – (7) $\Rightarrow$ $$-2\,ct^S = (\lambda-\mu)\,x^B - (\lambda+\mu)\,ct^B \qquad \Rightarrow$$

$\Rightarrow$ $$-ct^S = \frac{\lambda-\mu}{2}\,x^B - \frac{\lambda+\mu}{2}\,ct^B \qquad \Rightarrow$$

$\Rightarrow$ $$-t^S = b\,x^B - a\,ct^B \qquad \Rightarrow$$

$\Rightarrow$ $$t^S = a\,ct^B - b\,x^B \qquad \text{repeats (10)}$$

Workout (11)

(9)(x^S=0) $\Rightarrow$ $$0 = ax^B - bct^B \qquad \Rightarrow$$

$\Rightarrow$ $$v := \frac{x^B}{t^B} = \frac{a}{bc}$$

Section 2, requirement I: *proportionality under v* means that *v* should be the same over all measurements (in the line of the relative movement of S and B, but this can be generalized), hence everywhere equal to the value of v that we get if we measure it at x^S=0, as done in the main text. The procedure should be symmetric (could just as well start from x^B=0)

Workout (24)

$$x^S = ax^B - bct^B \qquad \text{repeats (9)}$$

$$\text{(9)(18)(20)} \Rightarrow \quad x^S = \frac{x^B}{\sqrt{1-\frac{v^2}{c^2}}} - t^B c \sqrt{\frac{1}{\frac{c^2}{v^2}-1}} \quad \Rightarrow$$

$$\Rightarrow \quad x^S = \frac{x^B}{\sqrt{1-\frac{v^2}{c^2}}} - t^B v \frac{c}{v} \sqrt{\frac{1}{\frac{c^2}{v^2}-1}} \quad \Rightarrow$$

$$\Rightarrow \quad x^S = \frac{x^B}{\sqrt{1-\frac{v^2}{c^2}}} - t^B v \sqrt{\frac{1}{\frac{v^2}{c^2}\left(\frac{c^2}{v^2}-1\right)}} \quad \Rightarrow$$

$$\Rightarrow \quad x^S = \frac{x^B}{\sqrt{1-\frac{v^2}{c^2}}} - t^B v \sqrt{\frac{1}{1-\frac{v^2}{c^2}}} \quad \Rightarrow$$

$$\Rightarrow \quad x^S = \frac{x^B - vt^B}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{repeats (24)}$$

Workout (24)

$$ct^S = act^B - bx^B \qquad \text{repeats (10)}$$

$$\Rightarrow \qquad t^S = at^B - \frac{b}{c}x^B \qquad (26)$$

$$(18)(20)(26) \Rightarrow \quad t^S = \frac{t^B}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{1}{c}\sqrt{\frac{1}{\frac{c^2}{v^2}-1}}\,x^B \quad \Rightarrow$$

$$\Rightarrow \quad t^S = \frac{t^B}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{1}{c}\frac{v}{c}\frac{c}{v}\sqrt{\frac{1}{\frac{c^2}{v^2}-1}}\,x^B \quad \Rightarrow$$

$$\Rightarrow \quad t^S = \frac{t^B}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{v}{c^2}\sqrt{\frac{1}{\frac{v^2}{c^2}\left(\frac{c^2}{v^2}-1\right)}}\,x^B \quad \Rightarrow$$

$$\Rightarrow \quad t^S = \frac{t^B}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\,x^B \quad \Rightarrow$$

$$\Rightarrow \quad t^S = \frac{t^B - \frac{v}{c^2}x^B}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \text{repeats (25)}$$